Isotropy Irreducible Varieties and Contact Geometry of Nilpotent Orbits

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Abstract

An isotropy irreducible variety is a homogeneous affine variety such that the isotropy representation is irreducible over $\mathbb{C}$. We study a relation between isotropy irreducible varieties and nilpotent orbits. If G/H is an isotropy irreducible variety and $\dim G > 1$, then G is semi-simple, hence its Lie algebra $\mathfrak{g}$ admits a natural decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an irreducible $\mathfrak{h}$-representation. First, we show that the highest weight orbit $O_\mathfrak{m} \subset \mathbb{P}(\mathfrak{m})$ is an integral submanifold of the natural contact structure of a nilpotent orbit $Z_\mathfrak{m} \subset \mathbb{P}(\mathfrak{g})$. Next, we focus on the case where $O_\mathfrak{m}$ is a Legendrian submanifold of $Z_\mathfrak{m}$, and study the associated Legendre moduli space. As a corollary, for a simple Lie algebra $\mathfrak{s}$, we classify equivariant Legendrian embeddings of rational homogeneous spaces into nilpotent orbits in $\mathbb{P}(\mathfrak{s})$. Finally, in the case where $Z_\mathfrak{m}$ is simply connected, we prove that $Z_\mathfrak{m}$ is the leaf space of an integrable distribution constructed from the contact structure of the projectivized cotangent bundle $\mathbb{P}T^*(G/H)$.

1 Introduction

We are working over $\mathbb{C}$, the field of complex numbers. Let G/H be a coset variety of a connected reductive group G and a reductive subgroup H (hence G/H is affine), and assume that G acts on G/H effectively. For the identity element $e \in G$, let $\mathfrak{g} := T_eG$ and $\mathfrak{h} := T_eH$ be the tangent algebras. Then we say that the variety G/H is an *isotropy irreducible variety of type* $(\mathfrak{g}, \mathfrak{h})$ if the tangent space $T_{e \cdot H}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$ is an irreducible $\mathfrak{h}$-representation (Definition 2.1). In this case, we also say that the pair $(\mathfrak{g}, \mathfrak{h})$ is an *isotropy irreducible pair*.

Note that similar definitions also make sense for a coset manifold of compact real Lie groups. Such real manifolds are called *(strongly) isotropy irreducible spaces* in the literature, and have been studied as manifolds carrying canonical invariant Riemannian metrics. Most remarkably, they are classified by Manturov [11] [13] [12], Wolf [18] and Krämer [8], and a classification of isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$ in our setting can be deduced from their results (Theorem 2.3).

Geometry of an isotropy irreducible variety G/H is particularly interesting when G/H is not symmetric. Indeed, the structure of symmetric varieties is now well-understood. We give constructions of non-symmetric G/H in Example 2.4, which cover all the cases where $\mathfrak{g}$ is of classical type.

By the classification of isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}$ is semi-simple whenever $\dim \mathfrak{g} > 1$. Thus we assume that $\mathfrak{g}$ is semi-simple. Then for the Killing form b of $\mathfrak{g}$, since $\mathfrak{h}$ is a reductive subalgebra, the restriction $b|_\mathfrak{h}$ on $\mathfrak{h}$ is non-degenerate. Therefore we have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathfrak{m} := \{v \in \mathfrak{g} : b(v, \mathfrak{h}) = 0\}$, the orthogonal complement of $\mathfrak{h}$. Since the restriction

$b|_{\mathfrak{m}}$ on $\mathfrak{m}$ is also non-degenerate, $\mathfrak{m}$ is a self-dual $\mathfrak{h}$-representation, which is isomorphic to the isotropy representation $\mathfrak{g}/\mathfrak{h}$.

We investigate a connection between isotropy irreducible varieties and nilpotent orbits (Definition 3.3), which are well-known examples of homogeneous contact manifolds. First, we associate a nilpotent orbit in $\mathbb{P}(\mathfrak{g})$ to G/H.

Theorem 1.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be an isotropy irreducible pair with $\dim \mathfrak{g} > 1$. For the highest weight orbit $O_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{m})$, we have the following:*

1. *$O_{\mathfrak{m}}$ is an integral submanifold of the natural contact structure of a nilpotent orbit $Z_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{g})$.*
2. *If $(\mathfrak{g}, \mathfrak{h})$ is symmetric, then $O_{\mathfrak{m}}$ is a Legendrian submanifold of $Z_{\mathfrak{m}}$. If $(\mathfrak{g}, \mathfrak{h})$ is not symmetric, then $O_{\mathfrak{m}}$ is Legendrian only if it is indicated in Table 1.*
3. *Assume that $\mathfrak{g}$ is simple. Then $Z_{\mathfrak{m}}$ is the minimal nilpotent orbit $Z_{long} := \mathbb{P}(\mathcal{O}_{min}) \subset \mathbb{P}(\mathfrak{g})$ unless $(\mathfrak{g}, \mathfrak{h})$ is one of (A_{2l-1}, C_l) $(l \geq 2)$, $(C_l, C_p \oplus C_{l-p})$ $(1 \leq p \leq l-1)$, $(\mathfrak{so}(l), \mathfrak{so}(l-1))$ $(l \geq 5)$, (F_4, B_4), (E_6, F_4), and (B_3, G_2).*

A full list of $Z_{\mathfrak{m}}$ is given in Remark 7.1 and Table 2-3.

As an immediate corollary, there are rational homogeneous spaces of Picard number 1 which are not Hermitian symmetric spaces but admit equivariant Legendrian embeddings, namely

$$\begin{aligned} \mathrm{OG}(3, \mathbb{C}^9) &\hookrightarrow Z_{\mathrm{long}} \subset \mathbb{P}(\mathfrak{so}(16)), \\ \mathrm{LG}(2, \mathbb{C}^{2l}) &\hookrightarrow Z_{[2^2, 1^{2l-4}]} \subset \mathbb{P}(\mathfrak{sl}(2l)) \quad (l > 2), \\ F_4/P_1 &\hookrightarrow Z_{2A_1} \subset \mathbb{P}(E_6). \end{aligned}$$

See Table 1 and Table 3.

We also give a geometric characterization of the case where $O_{\mathfrak{m}}$ is Legendrian submanifold of Z_{long}. Namely, when $\mathfrak{g}$ is simple, we show that $O_{\mathfrak{m}}$ is a Legendrian submanifold of Z_{long} if and only if $O_{\mathfrak{m}}$ is the scheme-theoretic intersection of $\mathbb{P}(\mathfrak{m})$ and Z_{long}. See Corollary 4.8.

On the other hand, by Theorem 1.1, an isotropy irreducible variety G/H with $\dim \mathfrak{g} > 1$ comes with a diagram of G-equivariant morphisms

$$\begin{array}{ccccc} & & \mathcal{C} := G \times_H O_{\mathfrak{m}} & & \\ & \overset{\phi}{\swarrow} & & \overset{\psi}{\searrow} & \\ G/H & & & & Z_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{g}). \end{array} \tag{1}$$

Here, ϕ is the natural projection from the G-principal bundle $\mathcal{C} = G \times_H O_{\mathfrak{m}} := \{(g, z) : g \in G, z \in O_{\mathfrak{m}}\}/(g, z) \sim (g \cdot h^{-1}, h \cdot z)$, $\forall h \in H$, and ψ is defined as follows: If $[g, z] \in \mathcal{C}$ is the point represented by $(g, z) \in G \times O_{\mathfrak{m}}$, we put $\psi([g, z]) := g \cdot z$, which is well-defined since $z \in O_{\mathfrak{m}} \subset Z_{\mathfrak{m}}$ and $Z_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{g})$ is an adjoint orbit.

In the case where $O_{\mathfrak{m}}$ is a Legendrian submanifold of $Z_{\mathfrak{m}}$, we show that the diagram (1) is the diagram associated to a *Legendre moduli space* (Theorem 5.2) in the sense of Merkulov [14]. In the following, a *Legendrian sub-flag variety* of a nilpotent orbit Z means an equivariant embedding of a rational homogeneous space into Z as a Legendrian submanifold (Definition 5.3).

Theorem 1.2. *Let $\mathfrak{s}$ be a semi-simple Lie algebra, S_{ad} the adjoint group, and $Z \subset \mathbb{P}(\mathfrak{s})$ a nilpotent orbit. Let O be a Legendrian sub-flag subvariety of Z, and define a coset variety $M := S_{ad}/Stab_{S_{ad}}(O)$ and its base point $o := e \cdot Stab_{S_{ad}}(O)$. Then the diagram*

$$\begin{array}{ccccc} & & \{(g \cdot o, z) \in M \times Z : z \in g \cdot O\} & & \\ & \swarrow & & \searrow & \\ M & & & & Z \end{array} \tag{2}$$

equipped with the natural projections defines an analytic family of compact Legendrian submanifolds of Z, which is complete and maximal (see Theorem 5.2). Moreover, if $\mathfrak{s}$ is simple, then for a Levi subalgebra $\mathfrak{l}$ of the Lie algebra of $Stab_{S_{ad}}(O)$, one of the following holds:

1. *M is an isotropy irreducible variety $S_{ad}/N_{S_{ad}}(\mathfrak{l})$ of type $(\mathfrak{s}, \mathfrak{l})$ and the diagram (2) coincides with the diagram (1) associated to $S_{ad}/N_{S_{ad}}(\mathfrak{l})$; or*
2. *M is an irreducible Hermitian symmetric space (of compact type). All possible cases are listed in Table 4.*

In both cases, O is a highest weight orbit of the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{s}$ with respect to the Killing form.

As a corollary of Theorem 1.1 and Theorem 1.2, a classification of Legendrian sub-flag varieties of nilpotent orbits in projectivized simple Lie algebras follows.

Corollary 1.3. *A Legendrian sub-flag variety of a nilpotent orbit in a projectivized simple Lie algebra is one of Legendrian $O_{\mathfrak{m}} \subset Z_{\mathfrak{m}}$ in Table 1, $O_{\mathfrak{m}} \subset Z_{\mathfrak{m}}$ in Table 2-3 and Legendrian $O \subset Z$ in Table 4.*

Finally, for arbitrary isotropy irreducible variety G/H with $\dim \mathfrak{g} > 1$, we show that the diagram (1) can be recovered from the natural contact structure of the projectivized cotangent bundle $\mathbb{P}T^*(G/H)$. Indeed, $\mathcal{C} = G \times_H O_{\mathfrak{m}}$ can be considered as a subbundle of $\mathbb{P}T^*(G/H) \simeq G \times_H \mathbb{P}(\mathfrak{m}^*)$ via the self-duality $\mathfrak{m} \simeq \mathfrak{m}^*$. If we restrict the contact structure of $\mathbb{P}T^*(G/H)$ over $\mathcal{C}$, then it is no more a contact structure, but instead, it contains null-spaces (Definition 6.1), i.e. the degeneracy loci of the Levi tensor. In the following theorem, we show that in the diagram (1), ψ defines a foliation compatible with the distribution of null-spaces:

Theorem 1.4. *Let G/H be an isotropy irreducible variety of type $(\mathfrak{g}, \mathfrak{h})$ with $\dim \mathfrak{g} > 1$. Consider the natural contact structure Θ of $\mathbb{P}T^*(G/H)$ (see Example 3.2). Then the morphism $\psi : \mathcal{C} \to Z_{\mathfrak{m}}$ in the diagram (1) satisfies the following properties:*

1. *$\Theta \cap T\mathcal{C} = (d\psi)^{-1}(D)$ where D is the contact structure of $Z_{\mathfrak{m}}$.*
2. *The vertical distribution $\ker d\psi$ on $\mathcal{C}$ coincides with the distribution $Null^{\Theta \cap T\mathcal{C}}$ of the null-spaces of $\Theta \cap T\mathcal{C}$.*
3. *The leaf space $\tilde{Z}_{\mathfrak{m}}$ of $Null^{\Theta \cap T\mathcal{C}}$ equipped with the quotient morphism $\tilde{\psi} : \mathcal{C} \to \tilde{Z}_{\mathfrak{m}}$ exists, provided that either*
   - (a) *$Z_{\mathfrak{m}}$ is simply connected. In this case, $\tilde{Z}_{\mathfrak{m}} = Z_{\mathfrak{m}}$ and $\tilde{\psi} = \psi$; or*
   - (b) *$Z_{\mathfrak{m}}$ is not simply connected but G/H is simply connected. In this case, $\pi_1(Z_{\mathfrak{m}}) = \mathbb{Z}/2\mathbb{Z}$, $\tilde{Z}_{\mathfrak{m}}$ is a universal cover of $Z_{\mathfrak{m}}$ and ψ is the composition of $\tilde{\psi}$ and the covering $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$. Furthermore, $\tilde{Z}_{\mathfrak{m}}$ can be constructed as follows:*
     - i. *If $\mathfrak{g}$ is not simple, then $\mathfrak{g} = \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)$ for a symplectic vector space V, $\tilde{Z}_{\mathfrak{m}}$ is the complement of $\mathbb{P}(V \oplus 0) \cup \mathbb{P}(0 \oplus V)$ in $\mathbb{P}(V \oplus V)$, and the covering $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$ is induced by the natural map $V \to Sym^2(V) \simeq \mathfrak{sp}(V)$, $v \mapsto v^2$.*
     - ii. *If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ can be embedded into a simple Lie algebra $\mathfrak{s}$ so that $\tilde{Z}_{\mathfrak{m}}$ is a Zariski open G-orbit in $Z_{long} \subset \mathbb{P}(\mathfrak{s})$ and the covering map $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$ is induced by the orthogonal projection $\mathfrak{s} \twoheadrightarrow \mathfrak{g}$.*

The pairs $(\mathfrak{g}, \mathfrak{h})$ with $Z_{\mathfrak{m}}$ simply connected are given in Proposition 4.1 and Corollary 4.11.

It is interesting to observe a connection between the previous theorems and the classification of *shared orbit pairs* due to Brylinski and Kostant [2] (see also [5, Example 2.7.a]). Here, a *shared orbit pair* means a pair $(\mathcal{O}_1, \mathcal{O}_2)$ of nilpotent orbits $\mathcal{O}_i \subset \mathfrak{g}_i$ $(i = 1, 2)$ for reductive Lie algebras $\mathfrak{g}_1 < \mathfrak{g}_2$ such that there is a G_1-equivariant finite morphism $\overline{\mathcal{O}_2} \to \overline{\mathcal{O}_1}$. In fact, by combining the classifications, the exceptions with $Z_{\mathfrak{m}} \neq Z_{\text{long}}$ in Theorem 1.1 can be characterized as isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple and having a shared orbit pair. Also in the last part of Theorem 1.4, the simple Lie algebra $\mathfrak{s}$ is chosen so that the Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{s}$ have a shared orbit pair $(\mathcal{O} \subset \mathfrak{g}, \mathcal{O}_{\min} \subset \mathfrak{s})$ satisfying $\mathbb{P}(\mathcal{O}) = Z_{\mathfrak{m}}$. See Corollary 4.11.

This paper is organized as follows. In Section 2, we review the classification of isotropy irreducible pairs. In Section 3, we recall the notion of the contact structure and some properties of the nilpotent orbits as contact manifolds. In Section 4, we prove Theorem 1.1, by studying the structure of $\mathfrak{m}$ case-by-case. In Section 5, we recall the notion of the Legendre moduli space and Merkulov's result, and prove Theorem 1.2. In Section 6, we explain how to compute the null-spaces Lie theoretically, and then prove Theorem 1.4. In Section 7, we summarize the classification of isotropy irreducible pairs and our classification of Legendrian sub-flag varieties in four tables. Our numbering of nodes of Dynkin diagrams is also explained, following [15].

Contents

2 Isotropy Irreducible Varieties

We are working in the holomorphic category. For example, every manifold is assumed to be a complex manifold, unless otherwise stated. A *variety* means a locally closed subset of an affine space or a projective space in the Zariski topology.

First, recall the definition and the classification of isotropy irreducible varieties.

Definition 2.1. Let G be a connected reductive group, G/H its coset variety by a reductive subgroup H, and H^0 the identity component of H.

1. We say that G/H is an *isotropy irreducible variety of type* (T_eG, T_eH) if the natural action of G on G/H is effective and the tangent space $T_{e \cdot H}(G/H)$ is an irreducible H^0-representation.
2. A pair $(\mathfrak{g}, \mathfrak{h})$ of reductive Lie algebras is called an *isotropy irreducible pair* if there is an isotropy irreducible variety G/H with $\mathfrak{g} = T_eG$ and $\mathfrak{h} = T_eH$.

Proposition 2.2. *Let $G_{\mathbb{R}}$ be a compact connected real Lie group and $H_{\mathbb{R}}$ a closed Lie subgroup. Assume that G and H be the complexifications of $G_{\mathbb{R}}$ and $H_{\mathbb{R}}$, respectively. Then the $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}/H_{\mathbb{R}}$ is effective if and only if so is the G-action on G/H.*

Proof. Recall that $H_{\mathbb{R}} = G_{\mathbb{R}} \cap H$, and so we can consider $G_{\mathbb{R}}/H_{\mathbb{R}}$ as a totally real submanifold of G/H.

Assume that the G-action on G/H is effective. Let $g \in G_{\mathbb{R}}$ be an element such that g acts trivially on $G_{\mathbb{R}}/H_{\mathbb{R}}$. Since the G-action on G/H is algebraic, the fixed-point-locus of g on G/H is Zariski closed. Since the fixed-point-locus contains $G_{\mathbb{R}}/H_{\mathbb{R}}$, it coincides with the whole variety G/H, a contradiction.

Assume that the $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}/H_{\mathbb{R}}$ is effective. Consider a subgroup of G defined by $K := \{g \in G : g \text{ acts on } G/H \text{ trivially}\}$. Then K is a normal closed subgroup of G and $K \cap G_{\mathbb{R}} = \{e\}$. Since G is reductive, so is K. Thus K is the complexification of its maximal compact subgroup, which is conjugate to a subgroup of $K \cap G_{\mathbb{R}} = \{e\}$. Hence $K = \{e\}$, i.e. the G-action on G/H is effective. □

Now a classification of isotropy irreducible pairs can be deduced from [18]. Indeed, in [18], there is a classification of closed subgroups $H_{\mathbb{R}}$ of a compact connected real Lie group $G_{\mathbb{R}}$ satisfying the following conditions:

1. the $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}/H_{\mathbb{R}}$ is effective;
2. if $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}$ are Lie algebras of $G_{\mathbb{R}}$ and $H_{\mathbb{R}}$, respectively, then $\mathfrak{g}_{\mathbb{R}}/\mathfrak{h}_{\mathbb{R}}$ is an $\mathfrak{h}_{\mathbb{R}}$-representation which is irreducible over $\mathbb{R}$; and
3. $G_{\mathbb{R}}/H_{\mathbb{R}}$ is simply connected.

By Proposition 2.2, our isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$ are corresponding to the complexifications of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$ in the classification in [18] such that $\mathfrak{g}_{\mathbb{R}}/\mathfrak{h}_{\mathbb{R}}$ is irreducible over $\mathbb{C}$.

Theorem 2.3 ([18, Theorem 11.1 and Correction])**.** *Let G/H be an isotropy irreducible variety of type $(\mathfrak{g}, \mathfrak{h})$ with $\dim \mathfrak{g} > 0$.*

1. *If G/H is symmetric, that is, there is a holomorphic involution $\theta : G \to G$ such that $\mathfrak{h}$ is the $(+1)$-eigenspace of $d_e\theta$, then one of the following holds:*
   (a) *$G = \mathbb{C}^\times$ and $H = \{e\}$;*
   (b) *$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^0$ and $\mathfrak{h} = diag(\mathfrak{g}^0)$ (i.e. $\mathfrak{h}$ is the diagonal) for some simple Lie algebra $\mathfrak{g}^0$; and*
   (c) *$\mathfrak{g}$ is simple.*
2. *If G/H is not symmetric, then $(\mathfrak{g}, \mathfrak{h})$ belongs to Table 1. In this case, $\mathfrak{g}$ is simple, $\mathfrak{h}$ is semi-simple and $rank(\mathfrak{h}) < rank(\mathfrak{g})$.*

Example 2.4. The following are examples of an embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ which defines a non-symmetric isotropy irreducible pair $(\mathfrak{g}, \mathfrak{h})$:

1. The adjoint representation $\mathfrak{h} \hookrightarrow \mathfrak{g} := \mathfrak{so}(\mathfrak{h})$ for simple $\mathfrak{h}$ not of type A ([18, Corollary 10.2]). In the numbering of Table 1, No. 15_n, 19_n, 21_n, 24, 26, 27, 28 and 29 correspond to the cases where $\mathfrak{h}$ is B_n, C_n, D_n, G_2, F_4, E_6, E_7 and E_8, respectively.
2. Isotropy representations of some rational homogeneous spaces L/P with L simple, P maximal parabolic and $\mathfrak{h}$ the semi-simple part of the Lie algebra of P. More precisely, there is the smallest nonzero P-invariant subspace T_1 in $T_{e\cdot P}(L/P)$, and by comparing [9, Proposition 2.6] and [18, Theorem 11.1 and Correction], we have the following examples:
   (a) $\mathfrak{h} \hookrightarrow \mathfrak{g} := \mathfrak{sl}(T_1)$ induced by an irreducible Hermitian symmetric space L/P, neither a projective space nor a quadric. In this case, $T_1 = T_{e\cdot P}(L/P)$. In the numbering of Table 1, No. $1_{p,q}$, 2, 3, 4_n and 5_n are the cases where L/P is $\mathrm{Gr}(q, \mathbb{C}^{p+q})$, $\mathbb{OP}^2$ (the Cayley plane), E_7/P_1 (the E_7-Hermitian symmetric space), $\mathbb{S}_n$ (the Spinor variety), and $\mathrm{LG}(n, \mathbb{C}^{2n})$ (the Lagrangian Grassmannian), respectively.

(b) $\mathfrak{h} \hookrightarrow \mathfrak{g} := \mathfrak{sp}(T_1)$ induced by an adjoint variety L/P for L not of type A, C (Definition 3.3). In the numbering of Table 1, No. 6, 7, 8, 9, 10 and 11_n are the cases where the Lie algebra of L is G_2, F_4, E_6, E_7, E_8 and $\mathfrak{so}(n+4)$, respectively.

(c) $\mathfrak{h} \hookrightarrow \mathfrak{g} := \mathfrak{so}(T_1)$ induced by the isotropy representation of $\mathrm{LG}(2, \mathbb{C}^{2n+4})$ $(n \geq 3)$, the isotropic Grassmannian of a symplectic vector space. Here, $\dim T_1 = 4n$ (and $\mathrm{codim} T_1 = 3$). This corresponds to No. 30_n in Table 1.

In particular, every non-symmetric isotropy pair $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ of type A or C is obtained in this way.

3. Complexification of isotropy representations of certain Riemannian symmetric spaces. Indeed, these cover all non-symmetric $(\mathfrak{g}, \mathfrak{h}) \neq (B_3, G_2)$ with $\mathfrak{g}$ classical, see [16] for a classification-free proof. For example, the previous examples in the item 2 can be obtained by taking
   (a) the compact presentation of the Hermitian symmetric space L/P,
   (b) the positive quaternionic-Kähler symmetric space of the same type with L, and
   (c) the quaternionic projective space $\mathbb{HP}^n$,

   respectively. See [16, Table 1, 2, 3].

4. The octonion representation $\mathfrak{h} := G_2 \hookrightarrow B_3 =: \mathfrak{g}$. This pair is No. 23 in Table 1, and described in detail in Section 4.3.

From now on, we assume that G/H is an isotropy irreducible variety of type $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ non-zero and semi-simple (or equivalently, $\dim \mathfrak{g} > 1$). Under this assumption, we fix our notation as follows. Let $G_{\mathrm{ad}} := G/Z(G)$ be the adjoint group of G. For the Killing form b of $\mathfrak{g}$, its restriction on $\mathfrak{h}$ is non-degenerate, and so $\mathfrak{m} := \{v \in \mathfrak{g} : b(v, \mathfrak{h}) = 0\}$ is a complementary subspace to $\mathfrak{h}$ in $\mathfrak{g}$. That is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as $\mathfrak{h}$-representations. The highest weight orbit is denoted by $O_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{m})$.

Next, we choose a maximal torus T_H of the identity component $H^0 \subset H$ with its Lie algebra $\mathfrak{t}_H (\leq \mathfrak{h})$. Then the weight decompositions of $\mathfrak{h}$ and $\mathfrak{m}$ are given as follows:

$$\mathfrak{h} = \mathfrak{t}_H \oplus \bigoplus_{\alpha \in R_{\mathfrak{h}}} \mathfrak{h}_\alpha, \quad \mathfrak{m} = \mathfrak{m}_0 \oplus \bigoplus_{w \in W} \mathfrak{m}_w.$$

Here, $R_{\mathfrak{h}}$ is the set of roots of $\mathfrak{h}$ and W is the set of nonzero weights of $\mathfrak{m}$. For $\alpha \in R_{\mathfrak{h}}$ and $w \in W \cup \{0\}$, $E_\alpha \in \mathfrak{h}_\alpha - \{0\}$ and $v_w \in \mathfrak{m}_w - \{0\}$ mean a root vector of $\mathfrak{h}$ and a weight vector of $\mathfrak{m}$, respectively. We also choose a Borel subgroup $B_H \subset H^0$ containing T_H, and its Lie algebra is written as $\mathfrak{b}_H$. The highest weight of $\mathfrak{m}$ (with respect to $\mathfrak{b}_H$) is denoted by $\rho \in W$ so that $O_{\mathfrak{m}} = H^0 \cdot [v_\rho] \subset \mathbb{P}(\mathfrak{m})$. For a simple Lie algebra $\mathfrak{h}_1$, its simple roots and the highest root are denoted by $\alpha_i^{\mathfrak{h}_1}$ and $\delta^{\mathfrak{h}_1}$, respectively, indexed as in Section 7, following [15]. If there is no ambiguity, we often omit the superscript $\mathfrak{h}_1$. Finally, we choose a maximal toral subalgebra $\mathfrak{t} \leq \mathfrak{t}_H \oplus \mathfrak{m}_0$. The set of roots of $\mathfrak{g}$ is also denoted by $R_{\mathfrak{g}}$.

Let us close this section with some observations.

Corollary 2.5. *1. ρ is a root of $\mathfrak{h}$ if and only if $\mathfrak{g}$ is not simple or $(\mathfrak{g}, \mathfrak{h})$ is one of (B_3, G_2), (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$ and (E_6, F_4).*

2. If $\mathfrak{g}$ is not simple, then $\rho = \delta$, the highest root of $\mathfrak{h}$.

3. If ρ is a root of $\mathfrak{h}$ and $\mathfrak{g}$ is simple, ρ is the dominant short root δ_{short} of $\mathfrak{h}$.

Proof. This is a direct consequence of Table 1 and the classification of symmetric isotropy irreducible varieties, which can be found in [17, (8.11.2) and (8.11.5)]. The latter is summarized in the first two columns of Table 2 and 3. □

Proposition 2.6. *1. $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$.*

2. For the normalizer $N_{G_{ad}}(\mathfrak{h})$ of $\mathfrak{h}$, the coset variety G_{ad}/N is an isotropy irreducible variety of type $(\mathfrak{g}, \mathfrak{h})$.

3. Under the quotient map $G \to G_{ad}$, the image of any closed subgroup of G with Lie algebra $\mathfrak{h}$ is contained in $N_{G_{ad}}(\mathfrak{h})$. In particular, there is a G-equivariant finite morphism $G/H \to G_{ad}/N_{G_{ad}}(\mathfrak{h})$.

Proof. 1. It follows from the irreducibility of $\mathfrak{g}/\mathfrak{h}$.

2. Since the Lie algebra of $N_{G_{\mathrm{ad}}}(\mathfrak{h})$ contains $\mathfrak{h}$, it is either $\mathfrak{h}$ or $\mathfrak{g}$. By Theorem 2.3, $\mathfrak{h} < \mathfrak{g}$ is not an ideal, and so its Lie algebra is $\mathfrak{h}$. Next, since G_{ad} is the adjoint group and since $\mathfrak{h}$ does not contain a simple factor of $\mathfrak{g}$, the G_{ad}-action on $G_{\mathrm{ad}}/N_{G_{\mathrm{ad}}}(\mathfrak{h})$ is effective.

3. It suffices to observe that every closed subgroup stabilizes its Lie algebra.

□

Corollary 2.7. *The stabilizer $Stab_G(O_{\mathfrak{m}})$ of $O_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{g})$ in G is the preimage of $N_{G_{ad}}(\mathfrak{h})$ under the quotient map $G \to G_{ad}$.*

Proof. Let $N := N_{G_{\mathrm{ad}}}(\mathfrak{h})$ and N^0 its identity component. By Proposition 2.6, it is enough to show that N stabilizes $O_{\mathfrak{m}}$. First, since N stabilizes $\mathbb{P}(\mathfrak{h})$, $\mathbb{P}(\mathfrak{m})$ is also N-stable, and so $g \cdot O_{\mathfrak{m}} \subset \mathbb{P}(\mathfrak{m})$ for $g \in N$. Since $N^0 \cdot (g \cdot O_{\mathfrak{m}}) = g \cdot (g^{-1} N^0 g) \cdot O_{\mathfrak{m}} = g \cdot (N^0 \cdot O_{\mathfrak{m}}) = g \cdot O_{\mathfrak{m}}$, $g \cdot O_{\mathfrak{m}}$ is a closed N^0-orbit, hence equal to $O_{\mathfrak{m}}$ by the irreducibility of $\mathfrak{m}$. □

3 Contact Geometry of Nilpotent Orbits

In this section, we recall the notion of the contact structure, and review contact geometry over nilpotent orbits.

Definition 3.1. Let Z be a manifold, and $D \subset TZ$ a holomorphic vector subbundle.

1. The *Levi tensor* Levi^D is a bundle morphism defined as

$$\mathrm{Levi}^D : \bigwedge^2 D \to TZ/D, \quad v \wedge w \mapsto [v, w] \mod D$$

   where v and w are local sections of D and $[v, w]$ denotes the Lie bracket of vector fields.

2. D is called a *contact structure* of Z if $D \subset TZ$ is of corank 1 and Levi^D_z is a non-degenerate 2-form on the fiber D_z for every $z \in Z$. In this case, Z is called a *contact manifold*, and the quotient line bundle $\mathcal{L} := TZ/D$ is called the *contact line bundle*.

3. A submanifold X of a contact manifold Z is called an *integral submanifold* of the contact structure if X is everywhere tangent to the contact structure. If furthermore $\dim Z = 2\dim X + 1$, we say that X is a *Legendrian* submanifold of Z.

Note that being a Legendrian submanifold means that its tangent space is a Lagrangian subspace of the contact structure at each point.

Example 3.2. Let Y be a manifold, and $Z := \mathbb{P}T^*Y$ its projectivized cotangent bundle. For $y \in Y$, each $z \in \mathbb{P}T^*_y Y \subset Z$ corresponds to a hyperplane $\mathrm{Ann}(z) \subset T_y Y$. If we define a hyperplane $\Theta_z \subset T_z Z$ as the preimage of $\mathrm{Ann}(z)$ under the differential $T_z Z \to T_y Y$ of the natural projection, then $\Theta := \bigcup_{z \in Z} \Theta_z$ becomes a contact structure of Z. Moreover, it is well-known that every Legendrian submanifold of Z can be obtained as the projectivized conormal bundle of a submanifold of Y.

From now on, we always denote by $\mathfrak{s}$ a semi-simple Lie algebra, and S_{ad} its adjoint group.

Definition 3.3. Let $\mathcal{N} \subset \mathfrak{s}$ be the cone of nilpotent elements.

1. An $S_{\rm ad}$-orbit contained in $\mathbb{P}(\mathcal{N})$ is called a *nilpotent orbit* of $\mathfrak{s}$.
2. If $\mathfrak{s}$ is simple, the $S_{\rm ad}$-orbit of a long root space in $\mathbb{P}(\mathfrak{s})$ is called the *adjoint variety* of $\mathfrak{s}$, and denoted by $Z_{\rm long}$.

Theorem 3.4 ([1, Remark 2.3])**.** *Let $Z \subset \mathbb{P}(\mathfrak{s})$ be a nilpotent orbit. Define a hyperplane subbundle $D \subset TZ$ as follows: for each $[v] \in Z$ and the stabilizer $\mathfrak{n}_{\mathfrak{s}}(v) := \{w \in \mathfrak{s} : [w, v] \in \mathbb{C} \cdot v\}$, put $D_{[v]} := v^{\perp}/\mathfrak{n}_{\mathfrak{s}}(v) \subset \mathfrak{s}/\mathfrak{n}_{\mathfrak{s}}(v) \simeq T_{[v]}Z$ where $v^{\perp}$ means the subspace consisting of elements orthogonal to v with respect to the Killing form. Then D is an S_{ad}-invariant contact structure of Z.*

Remark 3.5.

1. By a slight abuse of notation, we say that an $S_{\rm ad}$-orbit in the cone of nilpotent orbits $\mathcal{N} \subset \mathfrak{s}$ is a *nilpotent orbit* in $\mathfrak{s}$. For a nilpotent orbit $Z \subset \mathbb{P}(\mathfrak{s})$, its preimage $\mathcal{O} \subset \mathfrak{s}$ under the projection $\mathfrak{s} \smallsetminus \{0\} \to \mathbb{P}(\mathfrak{s})$ is a single nilpotent orbit, since every $S_{\rm ad}$-orbit in $\mathcal{N}$ is $\mathbb{C}^{\times}$-invariant. In this case, we write $Z = \mathbb{P}(\mathcal{O})$.
2. If $\mathfrak{s}$ is simple, then the adjoint variety $Z_{\rm long}$ is the highest weight orbit of the adjoint representation $\mathfrak{s}$, hence it is a unique closed $S_{\rm ad}$-orbit in $\mathbb{P}(\mathfrak{s})$. Similarly, its preimage $\mathcal{O}_{\rm min} \subset \mathfrak{s}$ under the projection $\mathfrak{s} \smallsetminus \{0\} \to \mathbb{P}(\mathfrak{s})$ is the minimal nilpotent orbit, in the sense that $\mathcal{O}_{\rm min}$ is contained in the closure of every nonzero nilpotent orbit in $\mathfrak{s}$.
3. The adjoint varieties are the only known examples of Fano contact manifold. In fact, it has been conjectured that every Fano contact manifold is isomorphic to an adjoint variety. We refer to [1] as a reference on this topic.

The following propositions are often useful.

Proposition 3.6. *Let $Z \subset \mathbb{P}(\mathfrak{s})$ be a nilpotent orbit. Then its contact line bundle is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathfrak{s})}(1)|_Z$.*

Proof. Write $Z = S_{\rm ad}/K$ for the stabilizer K of a point, say $[v] \in Z$. Then the tangent bundle of Z and its contact structure are given by $TZ \simeq S_{\rm ad} \times_K (\mathfrak{s}/\mathfrak{n}_{\mathfrak{s}}(v))$ and $D := S_{\rm ad} \times_K (v^{\perp}/\mathfrak{n}_{\mathfrak{s}}(v))$. Thus $\mathcal{L} := TZ/D \simeq S_{\rm ad} \times_K (\mathfrak{s}/v^{\perp})$. Observe that the Killing form induces a K-equivariant isomorphism $(\mathfrak{s}/v^{\perp}) \simeq (\mathbb{C} \cdot v)^*$, hence

$$\mathcal{L} \simeq S_{\rm ad} \times_K (\mathfrak{s}/v^{\perp}) \simeq S_{\rm ad} \times_K (\mathbb{C} \cdot v)^* \simeq \mathcal{O}_{\mathbb{P}(\mathfrak{s})}(1)|_Z.$$

□

Proposition 3.7. *Let $R \subset S_{ad}$ be a closed Lie subgroup. Let $w \in \mathfrak{s}$ be a nonzero nilpotent element, and $Z := S_{ad} \cdot [w] \subset \mathbb{P}(\mathfrak{s})$. Then the R-orbit $R \cdot [w]$ is an integral submanifold of the contact structure of Z if and only if w is orthogonal to the Lie algebra of R with respect to the Killing form of $\mathfrak{s}$.*

Proof. Recall that the contact structure of Z at $[w]$ is given by $w^{\perp}/\mathfrak{n}_{\mathfrak{s}}(w) \subset \mathfrak{s}/\mathfrak{n}_{\mathfrak{s}}(w) \simeq T_{[w]}Z$. Under this identification, the tangent space of $R \cdot [w]$ is $T_eR \mod \mathfrak{n}_{\mathfrak{s}}(w)$, and so it is contained in the contact hyperplane $w^{\perp}/\mathfrak{n}_{\mathfrak{s}}(w)$ if and only if $T_eR \subset w^{\perp}$. The statement follows since the contact structure of Z is $S_{\rm ad}$-invariant. □

Corollary 3.8. *Let $\mathfrak{r}$ be a reductive subalgebra of $\mathfrak{s}$ and $R \subset S_{ad}$ the associated connected Lie subgroup. Suppose that the $\mathfrak{r}$-representation $\mathfrak{s}/\mathfrak{r}$ does not have a common highest weight with the adjoint representation $\mathfrak{r}$. Then each positive dimensional highest weight R-orbit O in $\mathbb{P}(\mathfrak{s})$ outside $\mathbb{P}(\mathfrak{r})$ is an integral submanifold of the contact structure of a nilpotent orbit. Furthermore if rank$(\mathfrak{r})$ = rank$(\mathfrak{s})$, then O is contained in either Z_{long} or the orbit of short root spaces.*

Proof. Recall the orthogonal decomposition $\mathfrak{s} = \mathfrak{r} \oplus \mathfrak{r}^\perp$ with respect to the Killing form of $\mathfrak{s}$. Since $\mathfrak{r}^\perp (\simeq \mathfrak{s}/\mathfrak{r})$ and $\mathfrak{r}$ do not share an irreducible factor, every highest weight orbit in $\mathbb{P}(\mathfrak{s})$ not contained in $\mathbb{P}(\mathfrak{r})$ is in fact contained in $\mathbb{P}(\mathfrak{r}^\perp)$. Moreover, if $\mathrm{rank}(\mathfrak{r}) = \mathrm{rank}(\mathfrak{s})$, then the highest weight orbits must contain a root space of $\mathfrak{s}$. Thus by Proposition 3.7, it is enough to show that every positive dimensional highest weight orbit $O \subset \mathbb{P}(\mathfrak{s}) \smallsetminus \mathbb{P}(\mathfrak{r})$ is contained in a nilpotent orbit. To see this, consider a fixed point $[v] \in O$ of a Borel subgroup B_R of R. If we consider the decomposition $B_R = T_R \cdot U_R$ into a maximal torus T_R and the unipotent radical U_R, then T_R stabilizes the line $\mathbb{C} \cdot v (\subset \mathfrak{s})$, and U_R fixes the point $v (\in \mathfrak{s})$. If T_R fixes v, then the orbit $R \cdot v (\subset \mathfrak{s})$ is projective, a contradiction since O is not a point. Thus T_R acts on $\mathbb{C} \cdot v$ nontrivially, and by [1, Proposition 2.2], we conclude that v is an nilpotent element. □

By a similar argument, we associate a nilpotent orbit to each G/H.

Corollary 3.9. *In $\mathbb{P}(\mathfrak{g})$, $O_\mathfrak{m}$ is an integral submanifold of the contact structure of a nilpotent orbit $Z_\mathfrak{m}$.*

Proof. Since T_H acts non-trivially on $\mathfrak{m}_\rho$, a highest weight vector of $\mathfrak{m}$ is a nilpotent element by [1, Proposition 2.2]. If its adjoint orbit is denoted $Z_\mathfrak{m} \subset \mathbb{P}(\mathfrak{g})$, then by Proposition 3.7, $O_\mathfrak{m}$ is an integrabl submanifold of the contact structure of $Z_\mathfrak{m}$. □

Our notation for nilpotent orbits is as follows. As before, $\mathfrak{s}$ means a semi-simple Lie algebra, and S_{ad} is its adjoint group. When $\mathfrak{s}$ is simple, the S_{ad}-orbit of long (short, respectively) root spaces is denoted by $Z_{\mathrm{long}} \subset \mathbb{P}(\mathfrak{s})$ ($Z_{\mathrm{short}} \subset \mathbb{P}(\mathfrak{s})$, respectively). For each $(\mathfrak{g}, \mathfrak{h})$, the nilpotent orbit in Corollary 3.9 containing $O_\mathfrak{m}$ as an integral submanifold is denoted by $Z_\mathfrak{m} \subset \mathbb{P}(\mathfrak{g})$. For other nilpotent orbits in a projectivized simple Lie algebra, we use the labeling of nilpotent orbits described in [4]. Here is a brief explanation.

- If $\mathfrak{s}$ is a simple Lie algebra of classical type, then we consider its realization in terms of matrices, and label each nilpotent orbit by the Jordan type of matrices lying in the orbit. That is, if J_d is a $(d \times d)$ elementary Jordan matrix

$$J_d := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix} \quad (d \geq 2), \quad \text{and} \quad J_1 := (0),$$

then $Z_{[d_1, \cdots, d_k]}$ denotes a nilpotent orbit whose elements are represented by matrices conjugate to a Jordan matrix

$$\begin{pmatrix} J_{d_1} & & & \\ & J_{d_2} & & \\ & & \ddots & \\ & & & J_{d_k} \end{pmatrix}, \quad d_1 \geq \cdots \geq d_k \geq 1.$$

For example,

$$Z_{\mathrm{long}} = \begin{cases} Z_{[2, 1^{r-1}]} & \text{if } \mathfrak{s} = A_r = \mathfrak{sl}(r+1), \\ Z_{[2, 1^{2r-2}]} & \text{if } \mathfrak{s} = C_r = \mathfrak{sp}(2r), \\ Z_{[2^2, 1^{n-4}]} & \text{if } \mathfrak{s} = \mathfrak{so}(n), \end{cases}$$

and

$$Z_{\mathrm{short}} = \begin{cases} Z_{[2^2, 1^{2r-4}]} & \text{if } \mathfrak{s} = C_r = \mathfrak{sp}(2r), \\ Z_{[3, 1^{2r-2}]} & \text{if } \mathfrak{s} = B_r = \mathfrak{so}(2r+1). \end{cases}$$

See [4, §5.4].

- If $\mathfrak{s}$ is a simple Lie algebra of exceptional type, then we use the Bala-Carter classification, see [4, Ch. 8]. For example,

$$Z_{\text{long}} = Z_{A_1}, \quad \text{and} \quad Z_{\text{short}} = Z_{\tilde{A}_1}.$$

As another example, when $\mathfrak{s} = E_6$, for a nilpotent element $v \in \mathfrak{s}$ such that the semi-simple part of the smallest Levi subalgebra containing v is $A_1 \oplus A_1$, then the nilpotent orbit $S_{\text{ad}} \cdot [v]$ is denoted by Z_{2A_1}.

4 Structures of the Isotropy Representation: Proof of Theorem 1.1

In this section, we find $Z_{\mathfrak{m}}$ for each $(\mathfrak{g}, \mathfrak{h})$, i.e. the nilpotent orbit containing $O_{\mathfrak{m}}$.

4.1 The case where $\mathfrak{g}$ is not simple

First we consider the case where G/H is locally isomorphic to a group variety.

Proposition 4.1. *Assume that $\mathfrak{g}$ is not simple, i.e. $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^0$ and $\mathfrak{h} = diag(\mathfrak{g}^0)$ for some simple Lie algebra $\mathfrak{g}^0$. Let $\mathcal{O}_{min} \subset \mathfrak{g}^0$ be the minimal nilpotent orbit. Then the following hold:*

1. *$O_{\mathfrak{m}} = \mathbb{P}(\{(v \oplus (-v)) \in \mathfrak{g} : v \in \mathcal{O}_{min}\})$ and $Z_{\mathfrak{m}} = \mathbb{P}(\mathcal{O}_{min} \oplus \mathcal{O}_{min})$. In particular, $O_{\mathfrak{m}}$ is a Legendrian submanifold of $Z_{\mathfrak{m}}$.*
2. *If $\mathfrak{g}^0$ is not of type C, then $Z_{\mathfrak{m}}$ is simply connected.*
3. *If $\mathfrak{g}^0 = \mathfrak{sp}(V)$ for a $2r$-dimensional symplectic vector space V $(r \geq 1)$, then $\pi_1(Z_{\mathfrak{m}}) = \mathbb{Z}/2\mathbb{Z}$, and its universal covering is given by a morphism*

$$\begin{array}{rcl} \tilde{Z}_{\mathfrak{m}} := \mathbb{P}(V \oplus V) \setminus (\mathbb{P}(V \oplus 0) \cup \mathbb{P}(0 \oplus V)) & \to & Z_{\mathfrak{m}} \\ {[v \oplus w]} & \mapsto & [\nu(v) \oplus \nu(w)] \end{array}$$

where $\nu : V \to Sym^2(V) \simeq \mathfrak{g}^0$ is defined by $\nu(v) := v^2$.

Proof. Let G_0 be the adjoint group of $\mathfrak{g}^0$ so that $G_{\text{ad}} = G_0 \times G_0$. Then $O_{\mathfrak{m}}$ and $Z_{\mathfrak{m}}$ are homogeneous under the action of $\text{diag}(G_0)$ and $G_0 \times G_0$, respectively.

1. Let E_δ be the highest root vector of $\mathfrak{g}^0 (\simeq \mathfrak{h})$. Then since $\mathfrak{m} = \{x \oplus (-x) \in \mathfrak{g} : x \in \mathfrak{g}^0\}$, $O_{\mathfrak{m}}$ and $Z_{\mathfrak{m}}$ contain $[E_\delta \oplus (-E_\delta)]$, hence $O_{\mathfrak{m}} = \mathbb{P}(\{(v \oplus (-v)) \in \mathfrak{g} : v \in \mathcal{O}_{\min}\})$ and $Z_{\mathfrak{m}} = \mathbb{P}(\mathcal{O}_{\min} \oplus \mathcal{O}_{\min})$. Thus we have $\dim O_{\mathfrak{m}} = \dim \mathcal{O}_{\min} - 1$ and $\dim Z_{\mathfrak{m}} = 2 \dim \mathcal{O}_{\min} - 1$.
2. If $\mathfrak{g}^0$ is not of type C, then $\mathcal{O}_{\min}$ is simply connected by [4, Corollary 6.1.6 and §8.4], hence so is $Z_{\mathfrak{m}}$.
3. Suppose that $\mathfrak{g}^0 = \mathfrak{sp}(V)$ as in the statement. Then $\text{Sym}^2(V) \simeq \mathfrak{g}^0$ as $\mathfrak{g}^0$-representations, and the restriction of the natural map $\nu : V \to \text{Sym}^2(V) \simeq \mathfrak{g}^0$, $v \mapsto v^2$ defines a 2-to-1 covering $\nu_2 : V \setminus \{0\} \to \mathcal{O}_{\min}$. Since $\dim V = 2r \geq 2$, ν_2 is a universal covering. Thus a universal covering of $\mathcal{O}_{\min} \oplus \mathcal{O}_{\min}$ is given by a 4-to-1 covering

$$\nu_2 \oplus \nu_2 : (V \setminus \{0\}) \oplus (V \setminus \{0\}) \to \mathcal{O}_{\min} \oplus \mathcal{O}_{\min},$$

which induces a 2-to-1 covering from

$$\mathbb{P}((V \setminus \{0\}) \oplus (V \setminus \{0\})) = \mathbb{P}(V \oplus V) \setminus (\mathbb{P}(V \oplus 0) \cup \mathbb{P}(0 \oplus V)) = \tilde{Z}_{\mathfrak{m}}$$

to $\mathbb{P}(\mathcal{O}_{\min} \oplus \mathcal{O}_{\min}) = Z_{\mathfrak{m}}$. Finally, since $\mathbb{P}(V \oplus 0) \cup \mathbb{P}(0 \oplus V)$ is of codimension $2r(\geq 2)$ in $\mathbb{P}(V \oplus V)$, the covering $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$ is a universal covering.

□

4.2 The case where $\mathfrak{g}$ is simple

Now we consider the case where $\mathfrak{g}$ is simple. Let us begin with a simple observation.

Proposition 4.2. *Assume that ρ is not a root of $\mathfrak{h}$ (see Corollary 2.5). Then $\mathfrak{m}_\rho$ is a root space of $\mathfrak{g}$ with respect to $\mathfrak{t}$.*

Proof. Observe that since the ρ-weight space $\mathfrak{g}_\rho$ of $\mathfrak{g}$ (as a $\mathfrak{t}_H$-representation) is $\mathfrak{m}_\rho \oplus \mathfrak{h}_\rho$, if ρ is not a root of $\mathfrak{h}$, then $\mathfrak{m}_\rho = \mathfrak{g}_\rho$. Since $\mathfrak{t}_H \leq \mathfrak{t}$, a weight space of $\mathfrak{g}$ as a $\mathfrak{t}_H$-representation is generated by root spaces of $\mathfrak{g}$. Since $\mathfrak{g}_\rho$ is a highest weight space, it is of dimension 1, hence it coincides with a root space. □

Proposition 4.3. *Assume that $(\mathfrak{g}, \mathfrak{h})$ is symmetric and $\mathfrak{g}$ is simple.*

1. *If $(\mathfrak{g}, \mathfrak{h})$ is different from (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$ and (E_6, F_4), then $O_\mathfrak{m}$ is a Legendrian submanifold of $Z_\mathfrak{m}$. A list of $Z_\mathfrak{m}$ for such $(\mathfrak{g}, \mathfrak{h})$ is given in Table 2 (when rank$(\mathfrak{h})$ = rank$(\mathfrak{g})$) and Table 3 (when rank$(\mathfrak{h})$ < rank$(\mathfrak{g})$).*
2. *If $(\mathfrak{g}, \mathfrak{h})$ is one of (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$ and (E_6, F_4), then $Z_\mathfrak{m} \neq Z_{long}$.*

$Z_\mathfrak{m}$ for the exceptions are given in Section 4.3.

Proof of Proposition 4.3. First of all, if $(\mathfrak{g}, \mathfrak{h})$ is one of (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$ and (E_6, F_4), then since $2\dim O_\mathfrak{m} + 1 > \dim Z_{\text{long}}$, $Z_\mathfrak{m} \neq Z_{\text{long}}$

Thus we may assume that $(\mathfrak{g}, \mathfrak{h})$ is different from (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$ and (E_6, F_4). We need to find $Z_\mathfrak{m}$ and compare its dimension with $\dim O_\mathfrak{m}$. We use the well-known classification of symmetric varieties, which can be found in [17, (8.11.2) and (8.11.5)].

If rank$(\mathfrak{h})$ = rank$(\mathfrak{g})$, then $\mathfrak{h}$ is a maximal proper reductive subalgebra of maximal rank, hence $(\mathfrak{g}, \mathfrak{h})$ belongs to the first column of Table 2 (up to conjugacy). Moreover in this case, the highest weight ρ is indeed a root of $\mathfrak{g}$, which can be read off from [17, Theorem 8.10.9] and its proof. This information is summarized in the second column of Table 2. Since every root space of $\mathfrak{g}$ is 1-dimensional, $Z = Z_{\text{long}}$ (Z_{short}, respectively) if and only if ρ is long (short, respectively), hence the last column of Table 2 follow. The third column follows from [17, (8.11.2)], and by comparing the dimension of $O_\mathfrak{m}$ and Z, we conclude that $O_\mathfrak{m}$ is always Legendrian.

Next, assume that rank$(\mathfrak{h})$ < rank$(\mathfrak{g})$ so that $(\mathfrak{g}, \mathfrak{h})$ belongs to the first column of Table 3. The second and third columns also follow from [17, (8.11.5)]. By Corollary 2.5, the assumption implies that ρ is not a root of $\mathfrak{h}$, and since $\mathfrak{g}$ of type ADE, $O_\mathfrak{m} \subset Z_{\text{long}}$ by Proposition 4.2. Again by comparing the dimensions, we conclude that $O_\mathfrak{m}$ is a Legendrian submanifold of Z_{long}. □

Next, we focus on non-symmetric $(\mathfrak{g}, \mathfrak{h})$. Recall that this assumption implies that $\mathfrak{h}$ is semi-simple and rank$(\mathfrak{h})$ < rank$(\mathfrak{g})$ (Theorem 2.3). In particular, since $\mathfrak{t}_H \oplus \mathfrak{m}_0$ is the centralizer of $\mathfrak{t}_H$ in $\mathfrak{g}$, $\mathfrak{m}_0 \neq 0$. By the irreducibility of $\mathfrak{m}$, the $\mathfrak{h}$-representation generated by $\mathfrak{m}_0$ must be equal to $\mathfrak{m}$, so we see that W is contained in the root lattice $\mathbb{Z} \cdot R_\mathfrak{h}$ of $\mathfrak{h}$.

Lemma 4.4. *For each $w \in \mathbb{Q} \cdot R_\mathfrak{h}$, the $\mathbb{Q}$-vector space spanned by the roots of $\mathfrak{h}$, let $s(w) \in \mathbb{Z}$ be the sum of the coefficients in its expression with respect to the simple roots of $\mathfrak{h}$. If $W \subset \mathbb{Q} \cdot R_\mathfrak{h}$, then we have the following:*

1. *$\mathfrak{t}_H \oplus \mathfrak{m}_0 \oplus \bigoplus_{w \in W : s(w) = 0} \mathfrak{m}_w$ is a reductive subalgebra of $\mathfrak{g}$.*
2. *The vector subspace spanned by $\mathfrak{b}_H$, $\bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w$ and a Borel subalgebra of $\mathfrak{t}_H \oplus \mathfrak{m}_0 \oplus \bigoplus_{w \in W : s(w) = 0} \mathfrak{m}_w$ containing $\mathfrak{t}$ is a Borel subalgebra of $\mathfrak{g}$.*

Proof. 1. It is clear that $\mathfrak{k}_0 := \mathfrak{t}_H \oplus \mathfrak{m}_0 \oplus \bigoplus_{w \in W : s(w) = 0} \mathfrak{m}_w$ is a subalgebra of $\mathfrak{g}$. For algebraicity, observe that the subspace $\bigoplus_{w \in W : s(w) = 0} \mathfrak{m}_w$ is contained in the derived subalgebra $[\mathfrak{k}_0, \mathfrak{k}_0]$. Thus $\mathfrak{k}_0$ is generated by the algebraic subalgebras $[\mathfrak{k}_0, \mathfrak{k}_0]$ and $\mathfrak{t}_H \oplus \mathfrak{m}_0$, hence $\mathfrak{k}_0$ is also algebraic. Then $\mathfrak{k}_0$ is furthermore reductive since the restriction $b|_{\mathfrak{k}_0}$ of the Killing form is non-degenerate and by [15, Theorem 2, §1, Chapter 4].

2. Let $\mathfrak{b}_{\mathfrak{k}_0}$ be a Borel subalgebra of $\mathfrak{k}_0$ containing $\mathfrak{t}$, and put

$$\mathfrak{b} := (\mathfrak{b}_H + \mathfrak{b}_{\mathfrak{k}_0}) \oplus \bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w = \mathfrak{u}_H \oplus \mathfrak{b}_{\mathfrak{k}_0} \oplus \bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w$$

where $\mathfrak{u}_H$ is the unipotent radical of $\mathfrak{b}_H$. Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. Since $\mathfrak{u}_H \oplus \bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w$ is a solvable ideal of $\mathfrak{b}$, we see that $\mathfrak{b}$ is solvable.

To see the maximality of $\mathfrak{b}$, let $\tilde{\mathfrak{b}}$ be a solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ properly. Since $\mathfrak{b}$ contains $\mathfrak{t}$, both $\mathfrak{b}$ and $\tilde{\mathfrak{b}}$ are generated by root vectors of $\mathfrak{g}$. Thus there is a root $\beta \in R_{\mathfrak{g}}$ such that $\mathfrak{g}_\beta \smallsetminus \{0\} \subset \tilde{\mathfrak{b}} \smallsetminus \mathfrak{b}$. By its definition, $s(\beta|_{\mathfrak{t}_H}) \leq 0$.

- If $s(\beta|_{\mathfrak{t}_H}) = 0$, then $\mathfrak{g}_\beta \subset \mathfrak{k}_0$. Since $\mathfrak{b}_{\mathfrak{k}_0}$ is a Borel subalgebra of $\mathfrak{k}_0$, we have $\mathfrak{g}_\beta \subset \tilde{\mathfrak{b}} \cap \mathfrak{k}_0 = \mathfrak{b}_{\mathfrak{k}_0} \leq \mathfrak{b}$, a contradiction.
- If $s(\beta|_{\mathfrak{t}_H}) < 0$, then $\mathfrak{g}_{-\beta} \subset \mathfrak{b}$, hence the $\mathfrak{sl}(2)$-subalgebra $\mathfrak{g}_\beta \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \oplus \mathfrak{g}_{-\beta}$ is contained in $\tilde{\mathfrak{b}}$, a contradiction.

Therefore $\mathfrak{b}$ is a Borel subalgebra.

□

Before we proceed further, let us record another corollary of Table 1.

Corollary 4.5. *Assume that $(\mathfrak{g}, \mathfrak{h})$ is not symmetric. In the notation of Lemma 4.4, for the highest root $\delta^{\mathfrak{h}_1}$ of a simple factor $\mathfrak{h}_1$ of $\mathfrak{h}$, we have*

- $s(\rho) < s(\delta^{\mathfrak{h}_1})$ *if $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_1)$ is one of (B_3, G_2, G_2), $(E_7, A_1 \oplus F_4, F_4)$,*
- $s(\rho) = s(\delta^{\mathfrak{h}_1})$ *if $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_1)$ is one of $(D_{2n}, A_1 \oplus C_n, C_n)$ $(n \geq 3)$, $(F_4, A_1 \oplus G_2, G_2)$, $(E_6, A_2 \oplus G_2, G_2)$, $(E_8, G_2 \oplus F_4, F_4)$, and*
- $s(\rho) > s(\delta^{\mathfrak{h}_1})$ *otherwise.*

Proposition 4.6. *Assume that $(\mathfrak{g}, \mathfrak{h})$ is not symmetric. Then $\mathfrak{m}_\rho$ is a long root space of $\mathfrak{g}$, i.e. $Z_{\mathfrak{m}} = Z_{long}$, if and only if $(\mathfrak{g}, \mathfrak{h}) \neq (B_3, G_2)$.*

The non-symmetric exceptional case (B_3, G_2) is treated in Section 4.3.

Proof of Proposition 4.6. First of all, if $(\mathfrak{g}, \mathfrak{h}) = (B_3, G_2)$, then $O_{\mathfrak{m}} \notin Z_{\text{long}}$, since $2 \dim O_{\mathfrak{m}} + 1 = 11 > \dim Z_{\text{long}} = 7$.

Now we may assume that $(\mathfrak{g}, \mathfrak{h}) \neq (B_3, G_2)$. Then $\mathfrak{m}_\rho$ is a root space of $\mathfrak{g}$ with respect to $\mathfrak{t}$ by Proposition 4.2 and Corollary 2.5. In particular, if the Dynkin diagram of $\mathfrak{g}$ is simply laced, then $\mathfrak{m}_\rho$ is a long root space (with respect to $\mathfrak{t}$).

For the remaining cases, let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ constructed in Lemma 4.4. Note that for every $w \in W \cup \{0\}$ different from ρ, we have $s(w) < s(\rho)$. Moreover, by Corollary 4.5, if

- the Dynkin diagram of $\mathfrak{g}$ is not simply laced, and
- $(\mathfrak{g}, \mathfrak{h}) \neq (F_4, A_1 \oplus G_2)$,

then $s(\rho) > s(\delta)$, hence $\mathfrak{m}_\rho$ is $\mathfrak{b}$-stable. That is, $\mathfrak{m}_\rho$ is the highest root space of $\mathfrak{g}$ with respect to $\mathfrak{b}$. Therefore $\mathfrak{m}_\rho$ is a long root space if $(\mathfrak{g}, \mathfrak{h}) \neq (F_4, A_1 \oplus G_2)$.

Finally, assume that $(\mathfrak{g}, \mathfrak{h}) = (F_4, A_1 \oplus G_2)$. In this case, $s(\rho) = s(\delta)(= 5) > s(w)$ for the highest root δ of the G_2 factor and $w \in (R_{\mathfrak{h}} \smallsetminus \{\delta\}) \cup (W \smallsetminus \{\rho\})$. Moreover, since $\mathfrak{m}$ is isomorphic to the tensor product of an irreducible A_1-representation and a fundamental G_2-representation, each weight space $\mathfrak{m}_w$ is of dimension 1. Thus if $w \in W \smallsetminus R_{\mathfrak{h}}$, then $\mathfrak{m}_w$ is a root space of $\mathfrak{g}$. Now for $\mathfrak{k}_0 := \mathfrak{t}_H \oplus \mathfrak{m}_0 \oplus \bigoplus_{w \in W : s(w) = 0} \mathfrak{m}_w$ (as in Lemma 4.4),

$$[\mathfrak{k}_0, \mathfrak{h}_\delta \oplus \mathfrak{m}_\rho] = \mathfrak{h}_\delta \oplus \mathfrak{m}_\rho.$$

Let $\mathfrak{b}_{\mathfrak{k}_0}$ be a Borel subalgebra of $\mathfrak{k}_0$ containing $\mathfrak{t}$. If $[\mathfrak{b}_{\mathfrak{k}_0}, \mathfrak{m}_\rho] \subset \mathfrak{m}_\rho$, then $\mathfrak{m}_\rho$ is stable under the Borel subalgebra

$$(\mathfrak{b}_H + \mathfrak{b}_{\mathfrak{k}_0}) \oplus \bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w$$

of $\mathfrak{g}$ (Lemma 4.4). If $[\mathfrak{b}_{\mathfrak{k}_0}, \mathfrak{m}_\rho] \not\subset \mathfrak{m}_\rho$, then $\mathfrak{b}_{\mathfrak{k}_0}$ contains the $(\delta - \rho)$-weight space $\mathfrak{m}_{\delta-\rho}$, which is a root space since $(\delta - \rho) \notin R_{\mathfrak{h}}$. Then the opposite Borel subalgebra $\mathfrak{b}^-_{\mathfrak{k}_0} \leq \mathfrak{k}_0$ does not contain $\mathfrak{m}_{\delta-\rho}$, hence $\mathfrak{m}_\rho$ is stable under the Borel subalgebra

$$(\mathfrak{b}_H + \mathfrak{b}^-_{\mathfrak{k}_0}) \oplus \bigoplus_{w \in W : s(w) > 0} \mathfrak{m}_w$$

of $\mathfrak{g}$ (Lemma 4.4). □

In the setting of Proposition 4.6, the dimension of $O_\mathfrak{m}$ is given in Table 1. Note that $O_\mathfrak{m} \subset Z_{\text{long}}$ is not necessarily Legendrian.

Example 4.7. Here are some examples of non-symmetric $(\mathfrak{g}, \mathfrak{h})$ with $O_\mathfrak{m} \subset Z_{\text{long}}$ Legendrian.

1. For non-symmetric $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} = \mathfrak{sp}(2r)$, $r \geq 1$, $O_\mathfrak{m}$ is always a Legendrian submanifold of $Z_{\text{long}} \simeq \mathbb{P}^{2r-1}$. Indeed, this is called a *subadjoint variety* (Example 5.5) in the literature.
2. There are three infinite non-symmetric families with $O_\mathfrak{m} \subset Z_{\text{long}}$ Legendrian: $(A_{2p-1}, A_{p-1} \oplus A_1)$ $(p \geq 3)$, $(C_n, A_1 \oplus \mathfrak{so}(n))$ $(n \geq 3)$, and $(D_{2n}, A_1 \oplus C_n)$ $(n \geq 3)$. As in Examples 2.4, these are arising from the isotropy representations of $\mathrm{Gr}(2, \mathbb{C}^{p+2})$, $Z_{\mathfrak{so}(n+4)}$, and $\mathrm{LG}(2, \mathbb{C}^{2n+4})$, respectively. On the other hand, these three families are all possible non-symmetric $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ classical and $\mathfrak{h}$ not simple as proved in [18, Correction].

Before considering the exceptional cases, we show a geometric characterization of the cases where $O_\mathfrak{m} \subset Z_{\text{long}}$ as a Legendrian submanifold.

Corollary 4.8. *Assume that $\mathfrak{g}$ is simple and $Z_\mathfrak{m} = Z_{long}$, then the following are equivalent:*

1. *$O_\mathfrak{m}$ is a Legendrian submanifold of Z_{long},*
2. *for each $x \in O_\mathfrak{m}$, $T_x Z_{long} \cap T_x \mathbb{P}(\mathfrak{m}) = T_x O_\mathfrak{m}$ in $T_x \mathbb{P}(\mathfrak{g})$, and*
3. *$O_\mathfrak{m}$ is the scheme-theoretic intersection $Z_{long} \cap_{sch} \mathbb{P}(\mathfrak{m})$ in $\mathbb{P}(\mathfrak{g})$. That is, the ideal sheaf of $O_\mathfrak{m}$ is the sum of the ideal sheaves of Z_{long} and $\mathbb{P}(\mathfrak{m})$ in $\mathbb{P}(\mathfrak{g})$.*

Proof. Assume that $Z_\mathfrak{m} = Z_{\text{long}}$, i.e. $O_\mathfrak{m} \subset Z_{\text{long}}$. Then the condition (3) implies the condition (2) by [10, Lemma 5.1]. To see the converse implication (2) ⇒ (3), by the same lemma, it suffices to show that the condition (2) implies $O_\mathfrak{m} = Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})$ set-theoretically. Consider the inequalities

$$\dim(T_{[v_\rho]} Z_{\text{long}} \cap T_{[v_\rho]} \mathbb{P}(\mathfrak{m})) \geq \dim_{[v_\rho]}(Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})) \geq \dim O_\mathfrak{m}$$

where $\dim_{[v_\rho]}(Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m}))$ denotes the maximum among dimensions of irreducible components of $Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})$ containing $[v_\rho]$. Since $O_\mathfrak{m}$ is a unique closed H-orbit in $\mathbb{P}(\mathfrak{m})$ and Z_{long} is compact, it is contained in every irreducible component of $Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})$, hence $\dim_{[v_\rho]}(Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})) = \dim(Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m}))$. Therefore if the condition (2) holds, then $\dim O_\mathfrak{m} = \dim(Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m}))$. Then, since $O_\mathfrak{m}$ is compact and contained in every irreducible component of $Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})$, we see that $O_\mathfrak{m} = Z_{\text{long}} \cap \mathbb{P}(\mathfrak{m})$ set-theoretically, hence (2) ⇒ (3).

Next, we show the equivalence (1) ⇔ (2). Note that if ρ is a root of $\mathfrak{h}$, then by Corollary 2.5, Proposition 4.3 and Proposition 4.6, $Z_\mathfrak{m} \neq Z_{\text{long}}$, a contradiction. Thus the equivalence (1) ⇔ (2) follows from the inclusion $T_{[v_\rho]} Z_{\text{long}} \cap T_{[v_\rho]} \mathbb{P}(\mathfrak{m}) \supset T_{[v_\rho]} O_\mathfrak{m}$, the inequality

$$\dim Z_{\text{long}} + \dim \mathbb{P}(\mathfrak{m}) - \dim(T_{[v_\rho]} Z_{\text{long}} + T_{[v_\rho]} \mathbb{P}(\mathfrak{m})) = \dim(T_{[v_\rho]} Z_{\text{long}} \cap T_{[v_\rho]} \mathbb{P}(\mathfrak{m})) \geq \dim O_\mathfrak{m},$$

and the following lemma. □

Lemma 4.9. *Assume that $\rho \notin R_{\mathfrak{h}}$. Then*

$$\dim(T_{[v_\rho]}Z_{\mathfrak{m}} + T_{[v_\rho]}\mathbb{P}(\mathfrak{m})) = \dim \mathfrak{m} + \dim O_{\mathfrak{m}}$$

where the sum of the tangent spaces is taken in $T_{[v_\rho]}\mathbb{P}(\mathfrak{g})$.

Proof. If we identify $T_{[v_\rho]}\mathbb{P}(\mathfrak{g}) \simeq \mathfrak{g}/\mathfrak{m}_\rho$, then

$$T_{[v_\rho]}Z_{\mathfrak{m}} + T_{[v_\rho]}\mathbb{P}(\mathfrak{m}) = ([\mathfrak{g}, v_\rho] + \mathfrak{m})/\mathfrak{m}_\rho = \left(\sum_{w \in W} [\mathfrak{m}_w, v_\rho] + \mathfrak{m}\right)/\mathfrak{m}_\rho$$

since $[\mathfrak{h}, v_\rho] \subset \mathfrak{m}$ and $[\mathfrak{m}_0, \mathfrak{m}_\rho] \subset \mathfrak{m}_\rho$ (as ρ is not a root of $\mathfrak{h}$). For a weight vector $v_w \in \mathfrak{m}_w$, if $w = -\rho$, then the $\mathfrak{h}$-component of $[v_{-\rho}, v_\rho]$ is nonzero and spans $\mathbb{C} \cdot h_\rho$ where $h_\rho \in \mathfrak{t}_H$ is the $b|_{\mathfrak{t}_H}$-dual of ρ, i.e. $b(h_\rho, -) = \rho(-)$ on $\mathfrak{t}_H$, for the Killing form b of $\mathfrak{g}$. If $w \neq -\rho$, then the $\mathfrak{h}$-component of $[v_w, v_\rho]$ is contained in $\bigoplus_{\alpha \in R_{\mathfrak{h}}^+ : \alpha - \rho \in W} \mathfrak{h}_\alpha$, hence

$$\sum_{w \in W} [\mathfrak{m}_w, v_\rho] + \mathfrak{m} \subset \mathbb{C} \cdot h_\rho \oplus \bigoplus_{\alpha \in R_{\mathfrak{h}}^+ : \alpha - \rho \in W} \mathfrak{h}_\alpha \oplus \mathfrak{m}.$$

To show the converse inclusion, observe that for $\alpha \in R_{\mathfrak{h}}^+$, since $\rho + \alpha \notin W$, $\rho - \alpha \in W$ (equivalently, $\alpha - \rho \in W$) if and only if α is not orthogonal to ρ. Thus the number of $\alpha \in R_{\mathfrak{h}}^+$ such that $\rho - \alpha \in W$ is equal to $\dim O_{\mathfrak{m}}$. Furthermore, for $\alpha \in R_{\mathfrak{h}}^+$ satisfying $\rho - \alpha \in W$, we have $[v_\rho, E_{-\alpha}] \neq 0$, and so there is $v_{\alpha - \rho} \in \mathfrak{m}_{\alpha - \rho}$ such that the $\mathfrak{h}$-component of $[v_{\alpha-\rho}, v_\rho]$ is nonzero, since

$$b([v_{\alpha-\rho}, v_\rho], E_{-\alpha}) = b(v_{\alpha-\rho}, [v_\rho, E_{-\alpha}])$$

and since b is non-degenerate. Therefore $\bigoplus_{\alpha \in R_{\mathfrak{h}}^+ : \alpha - \rho \in W} \mathfrak{h}_\alpha \subset \sum_{w \in W} [\mathfrak{m}_w, v_\rho] + \mathfrak{m}$.

To summarize, we have

$$T_{[v_\rho]}Z_{\mathfrak{m}} + T_{[v_\rho]}\mathbb{P}(\mathfrak{m}) = \left(\mathbb{C} \cdot h_\rho \oplus \bigoplus_{\alpha \in R_{\mathfrak{h}}^+ : \alpha - \rho \in W} \mathfrak{h}_\alpha \oplus \mathfrak{m}\right)/\mathfrak{m}_\rho,$$

and its dimension is equal to

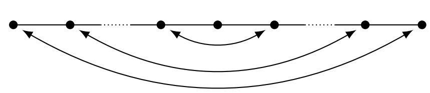
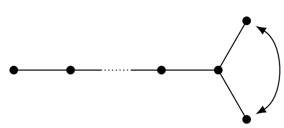
$$(1 + \dim O_{\mathfrak{m}} + \dim \mathfrak{m}) - 1.$$

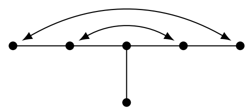
□

4.3 Exceptional Cases

In this section, we consider the exceptions in Proposition 4.3 and Proposition 4.6: (A_{2l-1}, C_l) $(l \geq 2)$, (D_{p+1}, B_p) $(p \geq 2)$, (E_6, F_4) (symmetric) and (B_3, G_2) (non-symmetric). We use their constructions in terms of diagram folding, which are given in [6, Example 2 and Theorem 5.15, §5, Ch. X].

First, suppose that $(\mathfrak{g}, \mathfrak{h})$ is one of the exceptional pairs which are symmetric. Consider a diagram automorphism of order 2 on the Dynkin diagram of $\mathfrak{g}$, given by switching nodes as follows:

- (A_{2l-1}, C_l) $(l \geq 2)$: 
- (D_{p+1}, B_p) $(p \geq 2)$: 

- (E_6, F_4): 

By identifying the nodes connected by arrows so that each identified node represents a short simple root, we obtain the Dynkin diagram of $\mathfrak{h}$. Furthermore, it induces an outer involution of $\mathfrak{g}$ such that the fixed-point-locus is $\mathfrak{h}$, and $\mathfrak{t}$ is stable under the involution.

To be precise, let us denote simple roots of $\mathfrak{h}$ and $\mathfrak{g}$ by α_i and β_i (labeled as in Section 7). If the nodes corresponding to β_i and β_j are connected by an arrow and folded to a node corresponding to α_k, then $\beta_i|_{\mathfrak{t}_H} = \beta_j|_{\mathfrak{t}_H} = \alpha_k$. Here is a list of such triples:

- (A_{2l-1}, C_l) $(l \geq 2)$: $\beta_i|_{\mathfrak{t}_H} = \beta_{2l-i}|_{\mathfrak{t}_H} = \alpha_i$, $1 \leq i \leq l-1$.
- (D_{p+1}, B_p) $(p \geq 2)$: $\beta_p|_{\mathfrak{t}_H} = \beta_{p+1}|_{\mathfrak{t}_H} = \alpha_p$.
- (E_6, F_4): $\beta_i|_{\mathfrak{t}_H} = \beta_{6-i}|_{\mathfrak{t}_H} = \alpha_i$, $i = 1, 2$.

Now we consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, which gives $\mathfrak{g}_\rho = \mathfrak{h}_\rho \oplus \mathfrak{m}_\rho$, orthogonal decomposition of the ρ-weight space (as a $\mathfrak{t}_H$-representation). In those exceptions, ρ is always the dominant short root (Corollary 2.5), hence $\mathfrak{g}_\rho$ is of dimension 2. It means that $\mathfrak{g}_\rho$ is generated by two root spaces, associated to two roots γ_1 and γ_2 of $\mathfrak{g}$ such that $\gamma_i|_{\mathfrak{t}_H} = \rho$. These γ_i are given as follows:

- (A_{2l-1}, C_l) $(l \geq 2)$: $\gamma_1 := \beta_1 + \cdots + \beta_{2l-2}$ and $\gamma_2 := \beta_2 + \cdots + \beta_{2l-1}$.
- (D_{p+1}, B_p) $(p \geq 2)$: $\gamma_1 := \beta_1 + \cdots + \beta_{p-1} + \beta_p$ and $\gamma_2 := \beta_1 + \cdots + \beta_{p-1} + \beta_{p+1}$.
- (E_6, F_4): $\gamma_1 := \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5 + \beta_6$ and $\gamma_2 := \beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 + \beta_6$.

Note that $\mathfrak{g}$ is of type ADE, and so all of γ_i is long. It means that $\mathfrak{m}_\rho$ is generated by a linear combination of two long root vectors. By Proposition 4.3, $v_\rho = a_1 \cdot E_{\gamma_1} + a_2 \cdot E_{\gamma_2}$ for some $a_i \in \mathbb{C}^\times$.

- If $(\mathfrak{g}, \mathfrak{h}) = (A_{2l-1}, C_l)$ $(l \geq 2)$, $\mathfrak{g} = \mathfrak{sl}(2l)$ is identified with the algebra of traceless matrices. We may choose $\mathfrak{t}$ as the subalgebra of the diagonal matrices, and then $\beta_i = \epsilon_i - \epsilon_{i+1}$ where $\epsilon_i : \mathfrak{t} \to \mathbb{C}$ is the linear functional which assigns the ith entry. The roots of $\mathfrak{g}$ are given by $\epsilon_i - \epsilon_j$ $(1 \leq i \neq j \leq 2l)$, and their root spaces are generated by e_{ij}, the elementary matrix with a unique nonzero entry at the ith row and the jth column. Thus $v_\rho = a_1 \cdot E_{\gamma_1} + a_2 \cdot E_{\gamma_2} = a_1 e_{1,\,2l-1} + a_2 e_{2,\,2l}$, and it is easy to show that it is conjugate to a Jordan matrix
$$\begin{pmatrix} J_2 & & & & \\ & J_2 & & & \\ & & J_1 & & \\ & & & \ddots & \\ & & & & J_1 \end{pmatrix}.$$
Thus $[\mathfrak{m}_\rho] \in Z_{[2,\,2,\,1,\,\cdots,\,1]} = Z_{[2^2,\,1^{2l-4}]}$.
- For the case (D_{p+1}, B_p) $(p \geq 2)$, we may proceed as in the previous case. Instead, let us introduce more elementary argument.
Let $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(n+1), \mathfrak{so}(n))$, $n \geq 2$. We may consider $\mathfrak{g}$ as the algebra of skew-symmetric $(n+1) \times (n+1)$ matrices, and $\mathfrak{h}$ as the subalgebra of the first $n \times n$ minors. Then the Killing form of $\mathfrak{g}$ is given as the trace form, and so the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ is consisting of matrices of form
$$\begin{pmatrix} & & & & x_1 \\ & & & & x_2 \\ & & & & \vdots \\ & & & & x_n \\ -x_1 & -x_2 & \cdots & -x_n & 0 \end{pmatrix}.$$

Moreover, as an $\mathfrak{so}(n)$-representation, it is isomorphic to the standard one. Thus the highest weight orbit $O_\mathfrak{m}$, which is the smooth quadric defined by $\sum_i x_i^2 = 0$, contains an element

$$\begin{pmatrix} & & & & 1 \\ & & & & \sqrt{-1} \\ & & & & \vdots \\ & & & & 0 \\ -1 & -\sqrt{-1} & \cdots & 0 & 0 \end{pmatrix}$$

whose Jordan normal form is

$$\begin{pmatrix} J_3 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_1 \end{pmatrix}.$$

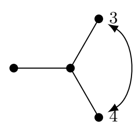
Thus $O_\mathfrak{m} \subset Z_{[3,\,1^{n-2}]}$ (which is equal to Z_{short} when n is even).

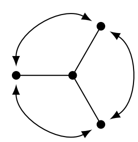
- If $(\mathfrak{g}, \mathfrak{h}) = (E_6, F_4)$, then for another root $\gamma_0 := \beta_1 + \beta_2 + 2\beta_3 + \beta_4 + \beta_5 + \beta_6$ and the reflection s_{γ_0} with respect to the hyperplane defined by γ_0, we have

$$s_{\gamma_0}(\beta_2) = \beta_2 + \gamma_0 = \gamma_2, \quad s_{\gamma_0}(\beta_4) = \beta_4 + \gamma_0 = \gamma_1.$$

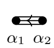
  This shows that $O_\mathfrak{m}$ is contained in the nilpotent orbit containing $[a_1 E_{\beta_4} + a_2 E_{\beta_2}]$, i.e. Z_{2A_1}.

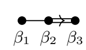
Next, assume that $(\mathfrak{g}, \mathfrak{h}) = (B_3, G_2)$, the only non-symmetric exception in Theorem 4.8. The embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ can be constructed as follows: Put $\tilde{\mathfrak{g}} := D_4$ and denote its simple roots by $\tilde{\beta}_1, \dots, \tilde{\beta}_4$. Consider the automorphisms σ_2 and σ_3 of $\tilde{\mathfrak{g}}$ induced by diagram automorphisms



and

, respectively.

Then $\mathfrak{g}$ and $\mathfrak{h}$ are fixed-point-loci of σ_2 and σ_3 in $\tilde{\mathfrak{g}}$, respectively. Thus we can choose $\mathfrak{t}$ and $\mathfrak{t}_H$ so that for the simple roots indexed as in the diagrams



and


,

we have

$$\tilde{\beta}_1|_\mathfrak{t} = \beta_1, \quad \tilde{\beta}_2|_\mathfrak{t} = \beta_2, \quad \tilde{\beta}_3|_\mathfrak{t} = \tilde{\beta}_4|_\mathfrak{t} = \beta_3,$$

and

$$\tilde{\beta}_2|_{\mathfrak{t}_H} = \alpha_2, \quad \tilde{\beta}_1|_{\mathfrak{t}_H} = \tilde{\beta}_3|_{\mathfrak{t}_H} = \tilde{\beta}_4|_{\mathfrak{t}_H} = \alpha_1.$$

Since $\rho = 2\alpha_1 + \alpha_2$, there are exactly two roots of $\mathfrak{g}$ whose restrictions on $\mathfrak{t}_H$ are equal to ρ, namely $\beta_1 + \beta_2 + \beta_3$ and $\beta_2 + 2\beta_3$. It means that $\mathfrak{m}_\rho$ is generated by $a_1 E_{\beta_1+\beta_2+\beta_3} + a_2 E_{\beta_2+2\beta_3}$ for some $a_i \in \mathbb{C}$. In fact, by [4, Remark 5.4.2, Theorem 5.1.2 and Corollary 6.1.4], there are 6 number of nilpotent orbits in $\mathbb{P}(\mathfrak{g})$

$$Z_{[7]}, \quad Z_{[5,\,1^2]}, \quad Z_{[3^2,\,1]}, \quad Z_{[3,\,2^2]}, \quad Z_{[3,\,1^4]}(= Z_{\text{short}}), \quad Z_{[2^2,\,1^3]}(= Z_{\text{long}})$$

of dimension

$$17, \quad 15, \quad 13, \quad 11, \quad 9, \quad 7,$$

respectively. Since $O_\mathfrak{m}(\simeq \mathbb{Q}^5)$ is of dimension 5, $Z_\mathfrak{m} \neq Z_{\text{short}}, Z_{\text{long}}$, hence $a_1, a_2 \neq 0$. Moreover, since the maximal torus of G acts on $\mathbb{P}(E_{\beta_1+\beta_2+\beta_3}, E_{\beta_2+2\beta_3}) \setminus \{[E_{\beta_1+\beta_2+\beta_3}], [E_{\beta_2+2\beta_3}]\}$ transitively, it is a subset of $Z_\mathfrak{m}$.

Now recall that $\mathfrak{g} = \mathfrak{so}(7)$ can be identified with the algebra of skew-symmetric 7×7 matrices. Following [6, §8, Ch. III], we consider $\mathfrak{t}$ generated by $h_i := e_{2i-1,\,2i} - e_{2i,\,2i-1}$ $(1 \le i \le 3)$ where e_{ij} denotes the elementary matrix with 1 at the intersection of the ith row and the jth column. Then the roots are given by $\pm\epsilon_i$ $(1 \le i \le 3)$, $\epsilon_i - \epsilon_j$ $(1 \le i \ne j \le 3)$ and $\pm(\epsilon_i + \epsilon_j)$ $(1 \le i < j \le 3)$ where ϵ_i is defined by setting $\epsilon_i(h_j) = -\sqrt{-1}\delta_{ij}$. For $\beta_1 := \epsilon_1 - \epsilon_2$, $\beta_2 := \epsilon_2 - \epsilon_3$ and $\beta_3 := \epsilon_3$, the root spaces associated to $\beta_1 + \beta_2 + \beta_3 (= \epsilon_1)$ and $\beta_2 + 2\beta_3 (= \epsilon_2 + \epsilon_3)$ are generated by matrices

$$\begin{pmatrix} &&&&&& 1 \\ &&&&&& -\sqrt{-1} \\ &&&&&& \\ &&&&&& \\ &&&&&& \\ &&&&&& \\ -1 & \sqrt{-1} &&&&& \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} &&&&&& \\ &&&&&& \\ &&&& 1 & -\sqrt{-1} & \\ &&&& -\sqrt{-1} & -1 & \\ && -1 & \sqrt{-1} &&& \\ && \sqrt{-1} & 1 &&& \\ &&&&&& \end{pmatrix},$$

respectively. Therefore $Z_{\mathfrak{m}}$ contains a point represented by a matrix

$$\begin{pmatrix} &&&&&& 1 \\ &&&&&& -\sqrt{-1} \\ &&&& 1 & -\sqrt{-1} & \\ &&&& -\sqrt{-1} & -1 & \\ && -1 & \sqrt{-1} &&& \\ && \sqrt{-1} & 1 &&& \\ -1 & \sqrt{-1} &&&&& \end{pmatrix}.$$

Its row rank is 4 and its 3rd power is zero, and so $Z_{\mathfrak{m}} = Z_{[3,\,2^2]}$.

In summary:

Proposition 4.10. *1. If $(\mathfrak{g}, \mathfrak{h}) = (A_{2l-1}, C_l)$ $(l \ge 2)$, then $Z_{\mathfrak{m}} = Z_{[2^2,\,1^{2l-4}]}$, a nilpotent orbit of dimension $8l - 9$.*

2. If $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(n+1), \mathfrak{so}(n))$ $(n \ge 2)$, then $Z_{\mathfrak{m}} = Z_{[3,\,1^{n-2}]}$, a nilpotent orbit of dimension $2n - 3$.

3. If $(\mathfrak{g}, \mathfrak{h}) = (E_6, F_4)$, then $Z_{\mathfrak{m}} = Z_{2A_1}$, a nilpotent orbit of dimension 31.

4. If $(\mathfrak{g}, \mathfrak{h}) = (B_3, G_2)$, then $Z_{\mathfrak{m}} = Z_{[3,\,2^2]}$, a nilpotent orbit of dimension 11.

In each case, $Z_{\mathfrak{m}} \ne Z_{long}$, and $O_{\mathfrak{m}} \subset Z_{\mathfrak{m}}$ is Legendrian.

4.4 Summary

Proof of Theorem 1.1. The first item follows from Corollary 3.9. The remaining parts follow from the full list of $Z_{\mathfrak{m}}$, which can be deduced from Proposition 4.1, Proposition 4.3, Proposition 4.6 and Proposition 4.10. □

As a corollary, we compute the fundamental group $\pi_1(Z_{\mathfrak{m}})$ of $Z_{\mathfrak{m}}$.

Corollary 4.11. *If $\mathfrak{g}$ is simple and $Z_{\mathfrak{m}}$ is not simply connected, then $(\mathfrak{g}, \mathfrak{h})$ is one of $(C_l, C_p \oplus C_{l-p})$ $(1 \le p \le l-1)$, $(\mathfrak{so}(l), \mathfrak{so}(l-1))$ $(l \ge 5)$, (F_4, B_4) (symmetric), and (B_3, G_2) (non-symmetric). In this case, $\pi_1(Z_{\mathfrak{m}}) = \mathbb{Z}/2\mathbb{Z}$, and its universal cover $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$ can be constructed as follows: $\mathfrak{g}$ can be embedded into a simple Lie algebra $\mathfrak{s}$ so that $\tilde{Z}_{\mathfrak{m}}$ is a Zariski open G_{ad}-orbit in $Z_{long} \subset \mathbb{P}(\mathfrak{s})$ and the covering map $\tilde{Z}_{\mathfrak{m}} \to Z_{\mathfrak{m}}$ is the restriction of the orthogonal projection $\mathfrak{s} \twoheadrightarrow \mathfrak{g}$.*

Proof. If $Z_{\mathfrak{m}} = Z_{\text{long}}$, then it is a rational homogeneous space, hence simply connected. By Theorem 1.1, it suffices to consider the following cases:

- $(A_{2l-1},\ C_l)$ $(l \geq 2)$: In this case, $Z_{\mathfrak{m}} = Z_{[2^2,\,1^{2l-4}]}$. If $l \geq 3$, then it is simply connected by [4, Corollary 6.1.6]. If $l = 2$, this pair coincides with $(D_3,\ B_2) = (\mathfrak{so}(6),\ \mathfrak{so}(5))$, which is considered below.
- $(E_6,\ F_4)$: In this case, $Z_{\mathfrak{m}} = Z_{2A_1}$, which is simply connected by [4, §8.4 Tables].
- $(C_l,\ C_p \oplus C_{l-p})$ $(1 \leq p \leq l-1)$, $(\mathfrak{so}(l),\ \mathfrak{so}(l-1))$ $(l \geq 5)$, $(F_4,\ B_4)$, and $(B_3,\ G_2)$: In these cases, $Z_{\mathfrak{m}}$ is $Z_{\text{short}}(= Z_{[2^2,\,1^{2l-4}]})$, $Z_{[3,\,1^{l-3}]}$, $Z_{\text{short}}(= Z_{\tilde{A}_1})$ and $Z_{[3,\,2^2]}$, respectively. Consider a simple Lie algebra

$$\mathfrak{s} := \begin{cases} A_{2l-1} & \text{if } (\mathfrak{g},\,\mathfrak{h}) = (C_l,\, C_p \oplus C_{l-p})\ (1 \leq p \leq l-1), \\ \mathfrak{so}(l+1) & \text{if } (\mathfrak{g},\,\mathfrak{h}) = (\mathfrak{so}(l),\, \mathfrak{so}(l-1))\ (l \geq 5), \\ E_6 & \text{if } (\mathfrak{g},\,\mathfrak{h}) = (F_4,\, B_4), \\ B_4 & \text{if } (\mathfrak{g},\,\mathfrak{h}) = (B_3,\, G_2). \end{cases}$$

For $(\mathfrak{g},\,\mathfrak{h}) \neq (B_3, G_2)$, consider the standard embedding $\mathfrak{g} \hookrightarrow \mathfrak{s}$. If $(\mathfrak{g},\,\mathfrak{h}) = (B_3, G_2)$, consider a non-standard embedding $\mathfrak{g} \hookrightarrow \mathfrak{s}$ defined as the composition $B_3 \hookrightarrow D_4 \xrightarrow{\sigma} D_4 \hookrightarrow B_4$ of the standard embeddings $B_3 \hookrightarrow D_4$ and $D_4 \hookrightarrow B_4$ and the triality σ. If $\mathcal{O} \subset \mathfrak{g}$ is the nilpotent orbit such that $\mathbb{P}\mathcal{O} = Z_{\mathfrak{m}}$ and $\mathcal{O}_{\min}$ is the minimal nilpotent orbit in $\mathfrak{s}$, then by [5, Proposition 2.12], the orthogonal projection $\mathfrak{s} \twoheadrightarrow \mathfrak{g}$ induces a finite G_{ad}-equivariant morphism $\overline{\mathcal{O}_{\min}} \twoheadrightarrow \overline{\mathcal{O}}$ between the closures, which is 2-to-1 over $\mathcal{O}$. This morphism induces a finite G_{ad}-equivariant morphism

$$\varphi : Z_{\text{long}}(= \mathbb{P}(\mathcal{O}_{\min}) \subset \mathbb{P}(\mathfrak{s})) \twoheadrightarrow \overline{Z_{\mathfrak{m}}}(= (\overline{\mathcal{O}} \setminus \{0\}/\mathbb{C}^{\times}) \subset \mathbb{P}(\mathfrak{g})),$$

which is 2-to-1 over $Z_{\mathfrak{m}}$. Furthermore, by the G_{ad}-equivariance, $\varphi^{-1}(Z_{\mathfrak{m}})$ is a Zariski open orbit in Z_{long}. Since nilpotent orbits are contact manifolds and $\overline{Z_{\mathfrak{m}}}$ is a union of nilpotent orbits, the complement of $Z_{\mathfrak{m}}$ in $\overline{Z_{\mathfrak{m}}}$ is of (complex) codimension at least 2. Thus $Z_{\text{long}} \setminus \varphi^{-1}(Z_{\mathfrak{m}})$ is of codimension at least 2 in Z_{long}, hence $\varphi^{-1}(Z_{\mathfrak{m}})$ is simply connected. It means that the restriction $\varphi^{-1}(Z_{\mathfrak{m}}) \to Z_{\mathfrak{m}}$ of φ is a universal cover, hence $\pi_1(Z_{\mathfrak{m}}) = \mathbb{Z}/2\mathbb{Z}$.

□

For the case where $\mathfrak{g}$ is not simple, see Proposition 4.1.

5 Legendre Moduli Spaces: Proof of Theorem 1.2

In this section, we recall the notion of Legendre moduli spaces, introduced by Merkulov in [14], and prove Theorem 1.2.

To state a precise definition, first we recall the construction of the Kodaira map associated to an analytic family of compact submanifolds, introduced by Kodaira in [7]. Suppose that Z is a manifold and

$$M \xleftarrow{\ p\ } \mathcal{X}(\subset M \times Z) \xrightarrow{\ q\ } Z$$

is a diagram of an analytic family of compact submanifolds of Z. That is, M is a connected manifold, $\mathcal{X}$ is a submanifold of $M \times Z$, and the natural projection p is a proper submersion with connected fibers. For each $t \in M$, put $\mathcal{X}_t := q(p^{-1}(t))$, the submanifold corresponding to the point t. Since p is proper, there are finitely many coordinate neighborhoods $U_i \subset Z$, $i \in I$, say with coordinate functions $(w_i^1, \ldots, w_i^c, z_i^1, \ldots, z_i^d)$, and a coordinate neighborhood $o \in U \subset M$ such that

- $\mathcal{X}_o \subset \bigcup_{i\in I} U_i$,
- for each $t \in U$ and $i \in I$, $\mathcal{X}_t \cap U_i$ is defined by a system of equations $w_i^\lambda = \varphi_i^\lambda(t, z_i^1, \ldots, z_i^d)$, $\forall\lambda = 1, \ldots, c$, and
- for each $i \in I$ and $\lambda = 1, \ldots, c$, φ_i^λ is a holomophic function on $U \times U_i$ satisfying $\varphi_i^\lambda|_{o\times U_i} = 0$.

Put $\varphi_i := (\varphi_i^1, \ldots, \varphi_i^c)$, a vector-valued function. Then for each tangent vector $\frac{\partial}{\partial t} \in T_o M$, a collection $\{\frac{\partial \varphi_i}{\partial t}\}_{i\in I}$ satisfies the cocycle condition for being a global section of the normal bundle $N_{\mathcal{X}_o/Z}$. Now the *Kodaira map* is defined to be a $\mathbb{C}$-linear map

$$\kappa : T_o M \to H^0(\mathcal{X}_o, N_{\mathcal{X}_o/Z}), \quad \frac{\partial}{\partial t} \mapsto \left\{\frac{\partial \varphi_i}{\partial t}\right\}_{i\in I}.$$

By the local nature of the Kodaira map, one can prove the following proposition:

Proposition 5.1. *Let $\mathcal{X} \to M$ and $\mathcal{X}' \to M'$ be analytic families of compact submanifolds of manifolds Z and Z', respectively. Fix two points $o \in M$ and $o' \in M'$, and suppose that*

1. *there is a holomorphic map $f : M \to M'$ with $f(o) = o'$, and*
2. *there exists a biholomorphism $F : U \to U'$ between open neighborhoods of $\mathcal{X}_o \subset Z$ and $\mathcal{X}'_{f(o)} \subset Z'$ such that $F(\mathcal{X}_t) = \mathcal{X}'_{f(t)}$ for all $t \in M$.*

Then there is a commutative diagram

$$\begin{array}{ccc} T_o M & \xrightarrow{\kappa} & H^0(\mathcal{X}_o, N_{\mathcal{X}_o/Z}) \\ \downarrow{\scriptstyle d_o f} & & \downarrow{\scriptstyle dF} \\ T_{f(o)} M' & \xrightarrow{\kappa'} & H^0(\mathcal{X}'_{f(o)}, N_{\mathcal{X}'_{f(o)}/Z'}) \end{array}$$

where the horizontal arrows are the Kodaira maps.

Now Merkulov's result can be stated as follows:

Theorem 5.2 ([14, Theorem 1.1])**.** *Let Z be a contact manifold with contact line bundle $\mathcal{L}$. Assume that X is a compact Legendrian submanifold of Z with $H^1(X, \mathcal{L}|_X) = 0$. Then there exists a manifold M equipped with a diagram*

$$M \xleftarrow{p} \mathcal{X}(\subset M \times Z) \xrightarrow{q} Z$$

of an analytic family of compact Legendrian submanifolds of Z containing X which is

1. *complete, i.e. for each $t \in M$, the composition of the Kodaira map and the projection*

$$T_t M \to H^0(\mathcal{X}_t, N_{\mathcal{X}_t/Z}) \to H^0(\mathcal{X}_t, \mathcal{L}|_{X_t})$$

is an isomorphism; and

2. *maximal, i.e. for each $t \in M$, if there is another analytic family $M' \xleftarrow{p'} \mathcal{X}' \xrightarrow{q'} Z$ of compact Legendrian submanifolds of Z with $t' \in M'$ satisfying $\mathcal{X}_t = \mathcal{X}'_{t'}$, then there exist an open neighborhood $t' \in U' \subset M'$ and a holomorphic function $f : U' \to M$ such that $f(t') = t$ and $\mathcal{X}_{f(t'')} = \mathcal{X}'_{t''}$ for all $t'' \in U'$.*

The manifold M is called the Legendre moduli space *associated to $X \subset Z$.*

From now on, to prove Theorem 1.2, we focus on Legendrian submanifolds of nilpotent orbits in $\mathbb{P}(\mathfrak{s})$. To simplify the notation, we introduce the following definitions:

Definition 5.3. Let $Z \subset \mathbb{P}(\mathfrak{s})$ be a nilpotent orbit and $\mathfrak{l} \subset \mathfrak{s}$ a reductive subalgebra.

1. A *highest weight* $\mathfrak{l}$*-orbit* in $\mathbb{P}(\mathfrak{s})$ means the highest weight orbit in $\mathbb{P}(V)$ for an irreducible $\mathfrak{l}$-subrepresentation $V \subset \mathfrak{s}$.
2. If a projective submanifold $O \subset \mathbb{P}(\mathfrak{s})$ is a highest weight $\mathfrak{l}$-orbit, we say that O is $\mathfrak{l}$-*homogeneous*. If furthermore O is contained in Z as a Legendrian submanifold, then we say that O is an *Legendrian* $\mathfrak{l}$*-sub-flag variety* of Z.

We often say that O is a *Legendrian sub-flag variety* of Z if $\mathfrak{l}$ is not specified in Definition 5.3. That is, a Legendrian sub-flag variety means an equivariant embedding of a rational homogeneous space as a Legendrian submanifold.

Remark 5.4. If a projective submanifold O of $\mathbb{P}(\mathfrak{s})$ is homogeneous under the action of a connected algebraic subgroup of S_{ad} with Lie algebra $\mathfrak{a}$, then it is a highest weight $\mathfrak{a}^{\mathrm{Levi}}$-orbit where $\mathfrak{a}^{\mathrm{Levi}}$ is a Levi subalgebra of $\mathfrak{a}$.

Example 5.5.
1. If $\mathfrak{s} = \mathfrak{sl}(r+1)$, $r \geq 1$, then $Z_{\mathrm{long}} \simeq \mathbb{P}T^*\mathbb{P}^r$. The contact structures as an adjoint variety (Theorem 3.4) and as a projectivized cotangent bundle (Example 3.2) coincide. If $\mathbb{P}^d \subset \mathbb{P}^r$ is a linear subspace of dimension d, then the projectivized conormal bundle $\mathbb{P}N^*_{\mathbb{P}^d/\mathbb{P}^r}$ is homogeneous under the action of $\mathrm{Stab}_{PGL(r+1)}(\mathbb{P}^d)$. Thus $\mathbb{P}N^*_{\mathbb{P}^d/\mathbb{P}^r}$ is a Legendrian $(D_1 \oplus \mathfrak{sl}(d+1) \oplus \mathfrak{sl}(r-d))$-sub-flag variety of Z_{long}. (Here, D_1 denotes a 1-dimensional toral subalgebra.) On the other hand, if $\mathbb{Q}^{r-1} \subset \mathbb{P}^r$ is a smooth quadric hypersurface, then $\mathbb{P}N^*_{\mathbb{Q}^{r-1}/\mathbb{P}^r} (\simeq \mathbb{Q}^{r-1})$ is a Legendrian $\mathfrak{so}(r+1)$-sub-flag variety of Z_{long}.
2. For an adjoint variety $Z_{\mathrm{long}} \subset \mathbb{P}(\mathfrak{s})$, the contact hyperplane D_o at the base point is a symplectic vector space. Moreover, the projectivization $\mathbb{P}(D_o)$ contains highest weight orbits with respect to the action the isotropy group, and each of them is a Legendrian submanifold of $\mathbb{P}(D_o)$. Such a Legendrian sub-flag variety is called a *subadjoint variety*. It is well-known that the subadjoint varieties are the only Legendrian sub-flag varieties of an odd dimensional projective space $\mathbb{P}^{2r-1}$, which is the adjoint variety for $\mathfrak{s} = \mathfrak{sp}(2r)$, $r \geq 1$, see [3, Theorem A.5] and the references therein. On the other hand, according to Table 1, every non-degenerate subadjoint variety can be obtained from a non-symmetric isotropy irreducible pair.

Proof of Theorem 1.2. Let O be a Legendrian sub-flag variety of a nilpotent orbit $Z \subset \mathbb{P}(\mathfrak{s})$. Let $\mathfrak{l}$ be a Levi subalgebra of the Lie algebra of $\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$ so that O is $\mathfrak{l}$-homogeneous. Consider the irreducible $\mathfrak{l}$-subrepresentation $V \subset \mathfrak{s}$ such that $O \subset \mathbb{P}(V)$ is the highest weight orbit. Since $\mathcal{L}|_O \simeq \mathcal{O}_{\mathbb{P}(V)}(1)|_O$ by Proposition 3.6, by the Bott-Borel-Weil theorem, $H^0(O, \mathcal{L}|_O) \simeq V^*$ while $H^q(O, \mathcal{L}|_O) = 0$, $\forall q \geq 1$. In particular, by Theorem 5.2, there exists the Legendre moduli space M' associated to O.

Next, as in the statement, put $o := e \cdot \mathrm{Stab}_{S_{\mathrm{ad}}}(O) \in M := S_{\mathrm{ad}}/\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$, and consider the diagram

$$\begin{array}{ccc} & \mathcal{X} := \{(g\cdot o, z) \in M \times Z \,:\, z \in g \cdot O\} & \\ \swarrow & & \searrow \\ M & & Z. \end{array}$$

In fact, with respect to the S_{ad}-action on $M \times Z$ (defined by $g \cdot (m, z) := (g \cdot m, g \cdot z)$), $\mathcal{X}$ is a single orbit containing $o \times O$, hence a submanifold of $M \times Z$. Since the morphism $\mathcal{X} \to M$ is a principal bundle with fiber $\simeq O$, the diagram defines an analytic family of compact Legendrian submanifolds of Z.

To prove that this family is complete and maximal, by homogeneity and by [14, Lemma 2.2], it is enough to show the completeness at $o \in M$. To see this, consider an open neighborhood $o \in U \subset M$ and a map $f : U \to M'$ with $o \mapsto [O]$, induced by the maximality of M'. Since $\mathcal{X}_s \neq \mathcal{X}_t$ for $s \neq t \in M$, f is injective. By Proposition 5.1, we have a commutative diagram

$$\begin{array}{ccccc} T_oU = T_oM & & & & \\ \downarrow{\scriptstyle d_o f} & \searrow^{\kappa} & & & \\ T_oM' & \xrightarrow{\kappa'} & H^0(O, N_{O/Z}) & \xrightarrow{r} & H^0(O, \mathcal{L}|_O) \simeq V^* \end{array}$$

where κ and κ' are the Kodaira maps, and κ is $\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$-equivariant. In particular, since V^* is an irreducible $\mathfrak{l}$-representation, the composition $r \circ \kappa$ is either zero or surjective. If it is zero, then since $r \circ \kappa'$ is an isomorphism, $d_o f$ is also zero, which is a contradiction since f is an injective holomorphic map. Therefore $f : U \to M'$ is a holomorphic injection of full rank, hence an open embedding. Thus the family is complete at $o \in M$.

Now assume that $\mathfrak{s}$ is simple. Then the S_{ad}-action on M is effective. Since $T_oM \simeq V^*$ as an $\mathfrak{l}$-representation, we see that the Lie algebra of $\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$ acts irreducibly on $T_oM = T_o(S_{\mathrm{ad}}/\mathrm{Stab}_{S_{\mathrm{ad}}}(O))$. It implies that its Lie algebra is a maximal subalgebra of $\mathfrak{s}$, hence either reductive or parabolic. In the former case, $\mathfrak{l}$ is indeed the Lie algebra of $\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$, hence M is an isotropy irreducible variety of type $(\mathfrak{s}, \mathfrak{l})$. Moreover, we have $V = \mathfrak{m}$ by Proposition 3.7, and $\mathrm{Stab}_{S_{\mathrm{ad}}}(O) = N_{S_{\mathrm{ad}}}(\mathfrak{l})$ by Corollary 2.7. The diagrams (1) and (2) coincide since an isotropy group of $\mathcal{X}$ is the isotropy group of $O(= O_{\mathfrak{m}})$. On the other hand, if $\mathrm{Stab}_{S_{\mathrm{ad}}}(O)$ is parabolic, then M is an irreducible Hermitian symmetric space since it is a rational homogeneous space whose isotropy representation is irreducible. The rest of the statement follows from the next proposition. □

Proposition 5.6. *Assume that $\mathfrak{s}$ is simple. Let $\mathfrak{l} < \mathfrak{s}$ be a maximal proper reductive subalgebra, and $O \subset \mathbb{P}(\mathfrak{s})$ a highest weight $\mathfrak{l}$-orbit not contained in $\mathbb{P}(\mathfrak{l})$. Assume that rank$(\mathfrak{l})$ = rank$(\mathfrak{s})$ but $(\mathfrak{s}, \mathfrak{l})$ is not an isotropy irreducible pair. Then we have the following:*

1. *$\mathfrak{s} \simeq \mathfrak{l} \oplus V \oplus V^*$ for a non-self-dual $\mathfrak{l}$-subrepresentation $V \subset \mathfrak{s}$, not necessarily irreducible.*
2. *O is a Legendrian submanifold of a nilpotent orbit $Z \subset \mathbb{P}(\mathfrak{s})$ if and only if $\mathfrak{l}$ is not semi-simple. In this case, the $\mathfrak{l}$-representation V is irreducible, $Z = Z_{long}$, the highest weight orbits $O^+ \subset \mathbb{P}(V)$ and $O^- \subset \mathbb{P}(V^*)$ are Legendrian submanifolds of Z_{long}, and O is one of them. Furthermore, the Lie algebras of $P^+ := Stab_{S_{ad}}(O^+)$ and $P^- := Stab_{S_{ad}}(O^-)$ are $\mathfrak{l} \oplus V$ and $\mathfrak{l} \oplus V^*$, respectively, and $S_{ad}/P^\pm$ are irreducible Hermitian symmetric spaces.*

All possible cases are listed in Table 4.

Proof. Observe that the weights of $\mathfrak{s}$ (as an $\mathfrak{l}$-representation) are roots of $\mathfrak{s}$, hence each weight space is of dimension 1. Thus there are only finitely many highest weight $\mathfrak{l}$-orbits, and by Corollary 3.8, each of them off $\mathbb{P}(\mathfrak{l})$, including O, is an integral submanifold of Z_{long} or Z_{short}.

Recall the classification of maximal proper reductive subalgebras of equal rank, given in [17, Theorem 8.10.9]. Such subalgebras $\mathfrak{l} < \mathfrak{s}$ which are not isotropy irreducible are listed in the first column of Table 4. Indeed, by [17, Theorem 8.10.9, (8.10.11), (8.10.14), and (8.10.15)], $\mathfrak{s} = \mathfrak{l} \oplus V \oplus V^*$ for a non-self-dual $\mathfrak{l}$-subrepresentation V, and V is irreducible if and only if $(\mathfrak{s}, \mathfrak{l}) \neq (E_8, A_4 \oplus A_4)$. (In fact, if $(\mathfrak{s}, \mathfrak{l}) = (E_8, A_4 \oplus A_4)$, then V is decomposed into two irreducible factors.) The highest weights of V are given in the second column of Table 4, where the third column also follows. Note that if a root α of $\mathfrak{s}$ is a highest weight of V and O is the corresponding highest weight orbit, then $O \subset Z_{\mathrm{long}}$ if α is long, and $O \subset Z_{\mathrm{short}}$ if α is short. Thus we obtain the fourth column. By comparing the dimensions, we determine when O is Legendrian, as indicated in the fifth column. Finally, observe that when it is Legendrian, then $Z = Z_{\mathrm{long}}$, and $\mathfrak{l} \oplus V$ and $\mathfrak{l} \oplus V^*$ are parabolic subalgebras defining Hermitian symmetric spaces, as observed in [17, p. 282]. Therefore the last column follows from the classification of irreducible Hermitian symmetric spaces. □

6 Distributions of Null-spaces: Proof of Theorem 1.4

Now we prove Theorem 1.4. The key ingredient is the natural contact structure of the projectivized cotangent bundle over G/H (Example 3.2).

Definition 6.1. Let Z be a complex manifold, and $D \subset TZ$ a hyperplane distribution. Then the *null-space* Null_z^D at $z \in Z$ is defined as the subspace of the fiber D_z consisting of null vectors with respect to Levi_z^D, i.e.

$$\mathrm{Null}_z^D := \{v \in D_z : \mathrm{Levi}_z^D(v, w) = 0, \ \forall w \in D_z\}.$$

Lemma 6.2. *Let L be a Lie group with Lie algebra $\mathfrak{l}$, and K its closed subgroup with Lie algebra $\mathfrak{k}$. Assume that there is an K-invariant subspace $\mathfrak{d} \subset \mathfrak{l}/\mathfrak{k}$, and define a vector subbundle $D := L \times_K \mathfrak{d} \subset L \times_K (\mathfrak{l}/\mathfrak{k}) \simeq T(L/K)$. For the quotient map $p : \mathfrak{l} \to \mathfrak{l}/\mathfrak{k}$ and $\tilde{\mathfrak{d}} := p^{-1}(\mathfrak{d})$, we have the following:*

1. *Under the identification $T_{e\cdot K}(L/K) \simeq \mathfrak{l}/\mathfrak{k}$, $\mathrm{Levi}_{e\cdot K}^D : \bigwedge^2 \mathfrak{d} \to \mathfrak{l}/\tilde{\mathfrak{d}}$ is given by*

$$\mathrm{Levi}_{e\cdot K}^D(v \mod \mathfrak{k}, w \mod \mathfrak{k}) = [v, w] \mod \tilde{\mathfrak{d}}, \quad \forall v, w \in \tilde{\mathfrak{d}}.$$

2. *The subspace $\mathfrak{null}^{\mathfrak{d}} := \{v \in \tilde{\mathfrak{d}} : [v, \tilde{\mathfrak{d}}] \subset \tilde{\mathfrak{d}}\}$ is a K-invariant subalgebra of $\mathfrak{l}$, containing $\mathfrak{k}$.*
3. *$\mathrm{Null}^D := \bigcup_{x \in L/K} \mathrm{Null}_x^D$ is an integrable vector subbundle of D over L/K, isomorphic to $L \times_K (\mathfrak{null}^{\mathfrak{d}}/\mathfrak{k})$.*
4. *For the connected Lie subgroup $\tilde{K}$ corresponding to $\mathfrak{null}^{\mathfrak{d}}$, the orbit of $e \cdot K$ is a leaf of Null^D, i.e. a maximal connected integral (immersed) submanifold. In particular, if $K \subset \tilde{K}$, then $L/\tilde{K}$ is the leaf space of Null^D and the natural projection $L/K \to L/\tilde{K}$ defines the associated foliation.*

Proof. 1. For each $g \in L$, let $\tau_g : L/K \to L/K$ be the left translation $x \mapsto g \cdot x$. Choose a vector subspace $W \subset \mathfrak{l}$ such that $\mathfrak{l} = W \oplus \mathfrak{k}$ as vector spaces. Let U be an open neighborhood of $0 \in W$ such that the exponential map $\exp : U \to L/K$, $y \mapsto \exp(y) \cdot K$ is an open embedding. Then for $X \in \mathfrak{l}/\mathfrak{k}$, define a (holomorphic) vector field V_X on $\exp(U) \cdot K$ as follows: for each $x \in \exp(U) \cdot K$ and a unique $y \in U$ such that $x = \exp(y) \cdot K$,

$$V_X(x) := d\tau_{\exp(y)}(X) \in T_x(L/K).$$

In other words, for a function f on U, we have

$$(V_X f)(x) = \frac{\partial}{\partial t} f(\exp(y) \exp(t\tilde{X}) \cdot K)\Big|_{t=0}$$

where $\tilde{X}$ is a unique element of W satisfying $p(\tilde{X}) = X$ and $t \in \mathbb{C}$ is a holomorphic parameter.

Claim. *For $X_1, X_2 \in \mathfrak{l}/\mathfrak{k}$, we have $[V_{X_1}, V_{X_2}](e \cdot K) = p([\tilde{X}_1, \tilde{X}_2])$.*

Assume the claim for a moment. Then for $X \in \mathfrak{d}$, since D is L-invariant, V_X is a local section of D extending X, and so the Levi tensor can be computed by using V_X. Thus for $X_1, X_2 \in \mathfrak{d}$, the claim implies $\mathrm{Levi}_{e\cdot K}^D(X_1, X_2) = [\tilde{X}_1, \tilde{X}_2] \mod \tilde{\mathfrak{d}}$, hence the statement follows.

Proof of Claim. The Lie bracket of vector fields is defined as $[V_{X_1}, V_{X_2}]f = V_{X_1}V_{X_2}f - V_{X_2}V_{X_1}f$ for a function f on $\exp(U)\cdot K$. At the base point $e\cdot K$, we have

$$\begin{aligned}(V_{X_1}V_{X_2}f)(e\cdot K) &= \frac{\partial}{\partial t}(V_{X_2}f)(\exp(t\tilde{X}_1)\cdot K)\bigg|_{t=0}\\ &= \frac{\partial^2}{\partial t\partial s}f(\exp(t\tilde{X}_1)\exp(s\tilde{X}_2)\cdot K)\bigg|_{s=t=0}\end{aligned}$$

for holomorphic parameters s and t near 0. From the formula, we see that for the vector fields $\tilde{X}_i^+$ on L/K generated by $\tilde{X}_i$, i.e.

$$\tilde{X}_i^+(x) := \frac{\partial}{\partial t}\exp(t\tilde{X}_i)\cdot x\bigg|_{t=0}, \quad \forall x \in L/K,$$

we have

$$[V_{X_1}, V_{X_2}](e\cdot K) = [\tilde{X}_2^+, \tilde{X}_1^+](e\cdot K).$$

Now observe that for $\tilde{X} \in \mathfrak{l}$, $\tilde{X}^+(e\cdot K) = p(\tilde{X})$, and it is well-known that $[\tilde{X}_1, \tilde{X}_2]^+ = -[\tilde{X}_1^+, \tilde{X}_2^+]$ (see [6, Theorem 3.4, §4, Ch. II]). □

2. For $v, w \in \mathfrak{null}^{\mathfrak{d}}$, we have $[v, w] \in \tilde{\mathfrak{d}}$ by the definition. For $X \in \tilde{\mathfrak{d}}$, since

$$[[v, w], X] = [[v, X], w] + [v, [w, X]] \in [\tilde{\mathfrak{d}}, w] + [v, \tilde{\mathfrak{d}}] \subset \tilde{\mathfrak{d}},$$

$[v, w] \in \mathfrak{null}^{\mathfrak{d}}$, hence $\mathfrak{null}^{\mathfrak{d}}$ is a subalgebra. The K-invariance follows from the invariance of $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$. Again, since $\tilde{\mathfrak{d}}$ is K-invariant, $\mathfrak{null}^{\mathfrak{d}}$ contains $\mathfrak{k}$.

3. Since D is L-invariant, each element of L sends a null-space to a null-space. Thus we have $\mathrm{Null}^D \simeq L\times_K \mathrm{Null}^D_{e\cdot K}$ in D. Now by the first statement, we have

$$\begin{aligned}\mathrm{Null}^D_{e\cdot K} &= \{v \in D_{e\cdot K} : \mathrm{Levi}^D_{e\cdot K}(v, w) = 0,\ \forall w \in D_{e\cdot K}\}\\ &= \{v \in \tilde{\mathfrak{d}} : [v, w] \in \tilde{\mathfrak{d}},\ \forall w \in \tilde{\mathfrak{d}}\}/\mathfrak{k}\\ &= \mathfrak{null}^{\mathfrak{d}}/\mathfrak{k}.\end{aligned}$$

Finally, Null^D is an integrable subbundle, since its Levi tensor is identically zero by its definition.

4. Since Null^D is an L-invariant integrable subbundle, L/K is the disjoint union of leaves of Null^D and each element of G sends a leaf to another leaf. Let $\mathcal{L}$ be the leaf of Null^D containing $e\cdot K$. Consider the quotient map $\pi : L \to L/K$. We claim that $\pi^{-1}(\mathcal{L})$ is a Lie subgroup of L. To see this, it is enough to show that it is a subgroup. Suppose that $g, h \in \pi^{-1}(\mathcal{L})$, i.e. $g\cdot K, h\cdot K \in \mathcal{L}$. Note that $g\cdot\mathcal{L}$ is a leaf containing $g\cdot K$, and so $g\cdot\mathcal{L} = \mathcal{L}$. It means $g\cdot h\cdot K \in \mathcal{L}$, hence $g\cdot h \in \pi^{-1}(\mathcal{L})$. It also means $g^{-1}\cdot\mathcal{L} = \mathcal{L}$, hence $g^{-1} \in \pi^{-1}(\mathcal{L})$. Therefore $\pi^{-1}(\mathcal{L})$ is a Lie subgroup of L, and the orbit of $e\cdot K$ under its action is $\mathcal{L}$. Moreover, since the orbit of $e\cdot K$ under the action of the Lie subgroup defined by $\mathfrak{null}^{\mathfrak{d}}$ is an integral submanifold, the tangent algebra of $\pi^{-1}(\mathcal{L})$ is $\mathfrak{null}^{\mathfrak{d}}$.

□

We identify the H-representations $\mathfrak{m}$ and $\mathfrak{m}^*$ via $b|_{\mathfrak{m}}$, the restriction of the Killing form. Also, G acts on $T^*(G/H)$ via $g\cdot\alpha := (\tau_{g^{-1}})^*(\alpha)$ ($g \in G$, $\alpha \in T^*(G/H)$, and $\tau_{g^{-1}}$ is the left translation by g^{-1}), which gives an isomorphism $\mathbb{P}T^*(G/H) \simeq G\times_H \mathbb{P}(\mathfrak{m}^*) \simeq G\times_H \mathbb{P}(\mathfrak{m})$ over G/H.

Lemma 6.3. *Define* $\mathcal{C} := G\times_H O_{\mathfrak{m}} \subset \mathbb{P}T^*(G/H)$, $v_\rho^* := b(v_\rho, -) \in \mathfrak{m}^* \simeq T^*_{e\cdot H}(G/H)$, *and* $\mathfrak{d} := \Theta_{[v_\rho^*]} \cap T_{[v_\rho^*]}\mathcal{C}$ *for the contact structure* Θ *of* $\mathbb{P}T^*(G/H)$ *described Example 3.2. Let* $G_{[v_\rho]} := \{g \in G : Ad_g(v_\rho) \in \mathfrak{m}_\rho\}$ *and* $H_{[v_\rho]} := H \cap G_{[v_\rho]}$ *be the stabilizers of the point* $[v_\rho] \in \mathbb{P}(\mathfrak{g})$.

1. *$\mathcal{C} = G \cdot [v_\rho^*] \simeq G/H_{[v_\rho]}$, and $\Theta \cap T\mathcal{C}$ is a well-defined vector bundle over $\mathcal{C}$, isomorphic to $G \times_{H_{[v_\rho]}} \mathfrak{d}$.*

2. *The Lie algebra of $G_{[v_\rho]}$ is $\mathfrak{null}^{\mathfrak{d}}$ (Lemma 6.2). In particular, for the natural quotient map $\psi : \mathcal{C} \simeq G/H_{[v_\rho]} \to G/G_{[v_\rho]} \simeq Z_{\mathfrak{m}}$, the vertical distribution $\ker d\psi$ over $\mathcal{C}$ coincides with $Null^{\Theta \cap T\mathcal{C}}$.*

3. *$\Theta \cap T\mathcal{C} = (d\psi)^{-1}(D)$ where D is the contact structure of $Z_{\mathfrak{m}}$.*

Proof. 1. $\mathcal{C}$ is G-homogeneous by its definition. The isomorphism $\mathfrak{m} \simeq \mathfrak{m}^*$ sends v_ρ to v_ρ^*, hence $\mathcal{C} = G \cdot [v_\rho^*] \simeq G/H_{[v_\rho]}$.

Next, since Θ and $T\mathcal{C}$ are G-invariant, so is $\Theta \cap T\mathcal{C}$. In particular, it is a subbundle of $T\mathcal{C}$, isomorphic to $G \times_{H_{[v_\rho]}} (\Theta_{[v_\rho^*]} \cap T_{[v_\rho^*]}\mathcal{C}) = G \times_{H_{[v_\rho]}} \mathfrak{d}$.

2. Put $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}_\rho) := \{X \in \mathfrak{g} : [X, \mathfrak{m}_\rho] \subset \mathfrak{m}_\rho\}$ and $\mathfrak{n}_{\mathfrak{h}}(\mathfrak{m}_\rho) := \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}_\rho) \cap \mathfrak{h}$, the Lie algebras of $G_{[v_\rho]}$ and $H_{[v_\rho]}$, respectively. Recall that the hyperplane in $T_{e \cdot H}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$ corresponding to $[v_\rho^*]$ is $v_\rho^\perp/\mathfrak{h}$ for $v_\rho^\perp := \{X \in \mathfrak{g} : b(X, v_\rho) = 0\}$. Thus $\mathfrak{d} := \Theta_{[v_\rho^*]} \cap T_{[v_\rho^*]}\mathcal{C} = v_\rho^\perp/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{m}_\rho)$ in $T_{[v_\rho^*]}\mathcal{C} \simeq \mathfrak{g}/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{m}_\rho)$. Therefore by the invariance of b,

$$\begin{aligned}
\mathfrak{null}^{\mathfrak{d}} &:= \{X \in v_\rho^\perp : [X, v_\rho^\perp] \subset v_\rho^\perp\} \\
&= \{X \in \mathfrak{g} : b(X, v_\rho) = 0, \text{ and } b([X, Y], v_\rho) = 0, \forall Y \in v_\rho^\perp\} \\
&= \{X \in \mathfrak{g} : b(X, [Y, v_\rho]) = 0, \forall Y \in v_\rho^\perp\} \quad (\because \mathfrak{t}_H \subset v_\rho^\perp) \\
&= \{X \in \mathfrak{g} : [X, v_\rho] \subset \mathbb{C} \cdot v_\rho\} \quad (\because b \text{ is non-degenerate}) \\
&= \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}_\rho).
\end{aligned}$$

The second statement follows from Lemma 6.2.

3. Previously, we have seen $\Theta \cap T\mathcal{C} = G \times_{H_{[v_\rho]}} (v_\rho^\perp/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{m}_\rho))$. Since $D = G \times_{G_{[v_\rho]}} (v_\rho^\perp/\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}_\rho))$, the statement follows.

□

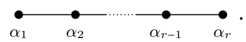
Proof of Theorem 1.4. Consider the natural map $\psi : \mathcal{C} \to Z_{\mathfrak{m}}$. The first two properties follow from Lemma 6.3. For the statements about the leaf space, first assume that $Z_{\mathfrak{m}}$ is simply connected. Then $G_{[v_\rho]}$ is connected, hence by Lemma 6.2, $\psi : \mathcal{C} \to Z_{\mathfrak{m}}$ defines the leaf space of $\mathrm{Null}^{\Theta \cap T\mathcal{C}}$. Thus we may assume that $Z_{\mathfrak{m}}$ is not simply connected but G/H is simply connected. Recall that $O_{\mathfrak{m}}$ is simply connected, hence so is $\mathcal{C}$. Thus we obtain a G-equivariant morphism $\tilde{\psi} : \mathcal{C} \to \tilde{Z}_{\mathfrak{m}}$ to the universal cover of $Z_{\mathfrak{m}}$, which is constructed in Proposition 4.1 and Corollary 4.11. As before, since $\tilde{Z}_{\mathfrak{m}}$ is G-homogeneous and its isotropy group is connected, $\tilde{Z}_{\mathfrak{m}}$ is the leaf space of $\mathrm{Null}^{\Theta \cap T\mathcal{C}}$ by Lemma 6.2. □

7 Tables

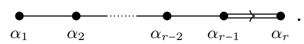
In this section, four tables are given. In Table 1, we recall the classification of $(\mathfrak{g}, \mathfrak{h})$, presented in [18, Section I.11 and Correction], together with dimension of $O_{\mathfrak{m}}$ and $Z_{\mathfrak{m}}$. In Table 2, Table 3, and Table 4, we collect Legendrian sub-flag varieties of nilpotent orbits, not listed in Table 1.

In the tables, we keep the notation of the previous sections. Namely, if $\mathfrak{h}$ is simple, then α_i, π_i and δ denote its simple root, its fundamental weight, and the highest root, respectively. For the indexing, we follow the notation of [15]. If $\mathfrak{h}$ has more than one simple factors and $\mathfrak{h}_1$ is one of them, then $\alpha_i^{\mathfrak{h}_1}$, $\pi_i^{\mathfrak{h}_1}$, and $\delta^{\mathfrak{h}_1}$ mean a simple root, a fundamental weight, and the highest root of $\mathfrak{h}_1$, respectively. To make the tables readable, for a simple Lie algebra which is not simply laced, we denote by δ_{short} its dominant short root. In our notation, δ and δ_{short} are given as follows:

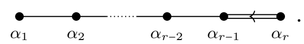
- A_r ($r \geq 1$): $\delta = \alpha_1 + \cdots + \alpha_r$ for the indexing



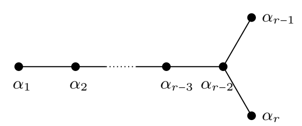
- B_r ($r \geq 2$): $\delta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$ and $\delta_{\text{short}} = \alpha_1 + \cdots + \alpha_r$ for the indexing



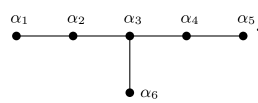
- C_r ($r \geq 2$): $\delta = 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r$ and $\delta_{\text{short}} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r$ for the indexing



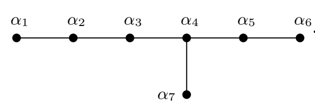
- D_r ($r \geq 3$): $\delta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ for the indexing



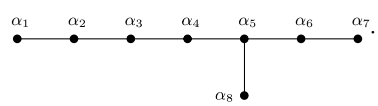
- E_6: $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ for the indexing



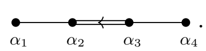
- E_7: $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$ for the indexing



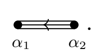
- E_8: $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ for the indexing



- F_4: $\delta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$ and $\delta_{\text{short}} = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ for the indexing



- G_2: $\delta = 3\alpha_1 + 2\alpha_2$ and $\delta_{\text{short}} = 2\alpha_1 + \alpha_2$ for the indexing



Remark 7.1. 1. $D_1 := \mathfrak{so}(2)$ is a 1-dimensional reductive Lie algebra.

2. In the isomorphism $D_2 := \mathfrak{so}(4) \simeq A_1 \oplus A_1$, the simple factors are written as A_1' and A_1''.

3. In the table of [18, Section I.11 and Correction], our non-symmetric isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$ are corresponding to (the complexifications of) the rows whose isotropy representations are irreducible over $\mathbb{C}$, i.e. diagrams in the column χ are connected. In the same table, the embedding of $\mathfrak{h}$ into $\mathfrak{g}$ is also described, in the column π.
4. For symmetric $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple, $Z_{\mathfrak{m}}$ is given in Table 2-3. For symmetric $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ not simple, then $Z_{\mathfrak{m}} = \mathbb{P}(\mathcal{O}_{\min} \oplus \mathcal{O}_{\min})$ in the notation of Proposition 4.1. For non-symmetric $(\mathfrak{g}, \mathfrak{h})$ (Table 1), $Z_{\mathfrak{m}} = Z_{\text{long}}$ if $(\mathfrak{g}, \mathfrak{h}) \neq (B_3, G_2)$ (Proposition 4.6) and $Z_{\mathfrak{m}} = Z_{[3, 2^2]}$ if $(\mathfrak{g}, \mathfrak{h}) = (B_3, G_2)$ (Proposition 4.10).

No.	$(\mathfrak{g},\,\mathfrak{h})$	Highest weight ρ of $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$	$\dim O_{\mathfrak{m}}$	$\dim Z_{\mathfrak{m}}$	Legendrian?
$1_{p,\,q}$ $(p \geq q \geq 2,\ pq > 4)$	$(A_{pq-1},\ A_{p-1} \oplus A_{q-1})$	$\pi_1^{A_{p-1}} + \pi_{p-1}^{A_{p-1}} + \pi_1^{A_{q-1}} + \pi_{q-1}^{A_{q-1}} = \delta^{A_{p-1}} + \delta^{A_{q-1}}$	$2p + 2q - 6$	$2pq - 3$	Yes if $q = 2$
2	$(A_{15},\ D_5)$	$\pi_4 + \pi_5 = \delta + \alpha_3 + \alpha_4 + \alpha_5$	14	29	Yes
3	$(A_{26},\ E_6)$	$\pi_1 + \pi_5 = \delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	24	51	
4_n $(n \geq 5)$	$(A_{n(n-1)/2-1},\ A_{n-1})$	$\pi_2 + \pi_{n-2} = \delta + \alpha_2 + \cdots + \alpha_{n-2}$	$4n - 12$	$n^2 - n - 3$	Yes if $n = 5$
5_n $(n \geq 3)$	$(A_{n(n+1)/2-1},\ A_{n-1})$	$2\pi_1 + 2\pi_{n-1} = 2\delta$	$2n - 3$	$n^2 + n - 3$	
6	$(C_2,\ A_1)$	$6\pi_1 = 3\delta$	1	3	Yes
7	$(C_7,\ C_3)$	$2\pi_3 = \delta + 2\alpha_2 + 2\alpha_3$	6	13	Yes
8	$(C_{10},\ A_5)$	$2\pi_3 = \delta + \alpha_2 + 2\alpha_3 + \alpha_4$	9	19	Yes
9	$(C_{16},\ D_6)$	$2\pi_5 = \delta + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	15	31	Yes
10	$(C_{28},\ E_7)$	$2\pi_1 = \delta + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7$	27	55	Yes
11_n $(n \geq 3)$	$(C_n,\ A_1 \oplus \mathfrak{so}(n))$	(if $n = 3$) $2\pi_1^{A_1} + 4\pi_1^{\mathfrak{so}(3)} = \delta^{A_1} + 2\delta^{\mathfrak{so}(3)}$ (if $n = 4$) $2\pi_1^{A_1} + 2\pi_1^{A_1'} + 2\pi_1^{A_1''} = \delta^{A_1} + \delta^{A_1'} + \delta^{A_1''}$ (if $n \geq 5$) $2\pi_1^{A_1} + 2\pi_1^{\mathfrak{so}(n)} = \delta^{A_1} + \delta^{\mathfrak{so}(n)} + \alpha_1^{\mathfrak{so}(n)}$	$n - 1$	$2n - 1$	Yes
12	$(D_{10},\ A_3)$	$\pi_1 + 2\pi_2 + \pi_3 = 2\delta + \alpha_2$	6	33	
13	$(D_{35},\ A_7)$	$\pi_3 + \pi_5 = \delta + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	21	133	
14	$(D_8,\ B_4)$	$\pi_3 = \delta + \alpha_3 + \alpha_4$	12	25	Yes
15_n $(n \geq 2)$	$(\mathfrak{so}(2n^2 + n),\ B_n)$	(if $n = 2$) $\pi_1 + 2\pi_2 = 2\delta - \alpha_2$ (if $n = 3$) $\pi_1 + 2\pi_3 = 2\delta - \alpha_2$ (if $n \geq 4$) $\pi_1 + \pi_3 = 2\delta - \alpha_2$	(if $n = 2$) 4 (if $n \geq 3$) $6n - 10$	$4n^2 + 2n - 7$	
16_n $(n \geq 2)$	$(\mathfrak{so}(2n^2 + 3n),\ B_n)$	(if $n = 2$) $2\pi_1 + 2\pi_2 = 2\delta + \alpha_1$ (if $n \geq 3$) $2\pi_1 + \pi_2 = 2\delta + \alpha_1$	$4n - 4$	$4n^2 + 6n - 7$	
17	$(D_{21},\ C_4)$	$2\pi_3 = \delta + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	12	77	
18_n $(n \geq 3)$	$(\mathfrak{so}(2n^2 - n - 1),\ C_n)$	$\pi_1 + \pi_3 = \delta + \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$	$6n - 10$	$4n^2 - 2n - 9$	
19_n $(n \geq 3)$	$(\mathfrak{so}(2n^2 + n),\ C_n)$	$2\pi_1 + \pi_2 = 2\delta - \alpha_1$	$4n - 4$	$4n^2 + 2n - 7$	
20	$(D_{64},\ D_8)$	$\pi_6 = \delta + \alpha_3 + 2\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7 + 2\alpha_8$	39	249	
21_n $(n \geq 4)$	$(\mathfrak{so}(2n^2 - n),\ D_n)$	(if $n = 4$) $\pi_1 + \pi_3 + \pi_4 = 2\delta - \alpha_2$ (if $n \geq 5$) $\pi_1 + \pi_3 = 2\delta - \alpha_2$	$6n - 13$	$4n^2 - 2n - 7$	
22_n $(n \geq 4)$	$(\mathfrak{so}(2n^2 + n - 1),\ D_n)$	$2\pi_1 + \pi_2 = 2\delta + \alpha_1$	$4n - 6$	$4n^2 + 2n - 9$	
23	$(B_3,\ G_2)$	$\pi_1 = \delta_{\text{short}}$	5	11	Yes
24	$(D_7,\ G_2)$	$3\pi_1 = 2\delta - \alpha_2$	5	21	
25	$(D_{13},\ F_4)$	$\pi_2 = \delta + \alpha_1 + 2\alpha_2 + \alpha_3$	20	45	
26	$(D_{26},\ F_4)$	$\pi_3 = 2\delta - \alpha_4$	20	97	
27	$(D_{39},\ E_6)$	$\pi_3 = 2\delta - \alpha_6$	29	149	
28	$(B_{66},\ E_7)$	$\pi_5 = 2\delta - \alpha_6$	47	259	
29	$(D_{124},\ E_8)$	$\pi_2 = 2\delta - \alpha_1$	83	489	
30_n $(n \geq 3)$	$(D_{2n},\ A_1 \oplus C_n)$	$2\pi_1^{A_1} + \pi_2^{C_n} = \delta^{A_1} + \delta_{\text{short}}^{C_n}$	$4n - 4$	$8n - 7$	Yes
31	$(G_2,\ A_1)$	$10\pi_1 = 5\delta$	1	5	
32	$(F_4,\ A_1 \oplus G_2)$	$4\pi_1^{A_1} + \pi_1^{G_2} = 2\delta^{A_1} + \delta_{\text{short}}^{G_2}$	6	15	
33	$(E_6,\ G_2)$	$\pi_1 + \pi_2 = \delta + \delta_{\text{short}}$	6	21	
34	$(E_6,\ A_2 \oplus G_2)$	$\pi_1^{A_2} + \pi_2^{A_2} + \pi_1^{G_2} = \delta^{A_2} + \delta_{\text{short}}^{G_2}$	8	21	
35	$(E_7,\ A_2)$	$4\pi_1 + 4\pi_2 = 4\delta$	3	33	
36	$(E_7,\ C_3 \oplus G_2)$	$\pi_2^{C_3} + \pi_1^{G_2} = \delta_{\text{short}}^{C_3} + \delta_{\text{short}}^{G_2}$	12	33	
37	$(E_7,\ A_1 \oplus F_4)$	$2\pi_1^{A_1} + \pi_1^{F_4} = \delta^{A_1} + \delta_{\text{short}}^{F_4}$	16	33	Yes
38	$(E_8,\ G_2 \oplus F_4)$	$\pi_1^{G_2} + \pi_1^{F_4} = \delta_{\text{short}}^{G_2} + \delta_{\text{short}}^{F_4}$	20	57	

Table 1: Classification of non-symmetric isotropy irreducible pairs $(\mathfrak{g}, \mathfrak{h})$.

$(\mathfrak{g},\, \mathfrak{h})$	Highest weight ρ of $\mathfrak{m}$ as a root of $\mathfrak{g}$	$\dim O_{\mathfrak{m}}$	$Z_{\mathfrak{m}}^{\dim Z_{\mathfrak{m}}}$
$(B_l,\, D_p \oplus B_{l-p})$ $(2 \le p \le l)$	$-\alpha_p$	(if $p < l$) $2l - 3$ (if $p = l$) $2l - 2$	(if $p < l$) Z_{long}^{4l-5} (if $p = l$) Z_{short}^{4l-3}
$(C_l,\, C_p \oplus C_{l-p})$ $(1 \le p \le l - 1)$	$-\alpha_p$	$2l - 3$	Z_{short}^{4l-3}
$(D_l,\, D_p \oplus D_{l-p})$ $(2 \le p \le l - 2)$	$-\alpha_p$	$2l - 4$	Z_{long}^{4l-7}
$(G_2,\, A_1 \oplus A_1)$	$-\alpha_2$	2	Z_{long}^{5}
$(F_4,\, C_3 \oplus C_1)$	$-\alpha_4$	7	Z_{long}^{15}
$(F_4,\, B_4)$	$-\alpha_1$	10	Z_{short}^{21}
$(E_6,\, A_5 \oplus A_1)$	one of $-\alpha_2,\, -\alpha_4,\, -\alpha_6$	10	Z_{long}^{21}
$(E_7,\, A_7)$	$-\alpha_7$	16	Z_{long}^{33}
$(E_7,\, D_6 \oplus A_1)$	one of $-\alpha_2,\, -\alpha_6$	16	Z_{long}^{33}
$(E_8,\, D_8)$	$-\alpha_7$	28	Z_{long}^{57}
$(E_8,\, E_7 \oplus A_1)$	$-\alpha_1$	28	Z_{long}^{57}

Table 2: Highest weight orbits for isotropy irreducible pairs $(\mathfrak{g},\, \mathfrak{h})$ of equal rank.

$(\mathfrak{g},\, \mathfrak{h})$	Highest weight ρ of $\mathfrak{m}$	$\dim O_{\mathfrak{m}}$	$Z_{\mathfrak{m}}^{\dim Z_{\mathfrak{m}}}$
$(A_{l-1},\, \mathfrak{so}(l))$ $(l \ge 4)$	(if $l = 4$) $2\pi_1^{A_1'} + 2\pi_1^{A_1''}$ (if $l \ge 5$) $2\pi_1^{\mathfrak{so}(l)}$	$l - 2$	Z_{long}^{2l-3}
$(A_{2l-1},\, C_l)$ $(l \ge 2)$	$\pi_2 = \delta_{\text{short}}$	$4l - 5$	$Z_{[2^2,\, 1^{2l-4}]}^{8l-9}$
$(D_{p+q+1},\, B_p \oplus B_q)$ $(p + q \ge 2,\, p \ge q \ge 0)$	(if $q > 0$) $\pi_1^{B_p} + \pi_1^{B_q}$ (if $q = 0$) $\pi_1^{B_p} = \delta_{\text{short}}$	(if $q > 0$) $2p + 2q - 2$ (if $q = 0$) $2p - 1$	(if $q > 0$) $Z_{\text{long}}^{4p+4q-3}$ (if $q = 0$) $Z_{[3,\, 1^{2p-1}]}^{4p-1}$
$(E_6,\, F_4)$	$\pi_1 = \delta_{\text{short}}$	15	$Z_{2A_1}^{31}$
$(E_6,\, C_4)$	π_4	10	Z_{long}^{21}

Table 3: Highest weight orbits for symmetric isotropy irreducible pairs $(\mathfrak{g},\, \mathfrak{h})$ of different rank.

$(\mathfrak{s},\, \mathfrak{l})$	Highest weights of V as roots of $\mathfrak{s}$	$\dim O$	$Z^{\dim Z}$	Legendrian?	$S_{\text{ad}}/P^{\pm}$
$(A_l,\, A_{p-1} \oplus A_{l-p} \oplus D_1)$ $(1 \le p \le l)$	$-\alpha_p$	$l - 1$	Z_{long}^{2l-1}	Yes	$\mathrm{Gr}(p,\, \mathbb{C}^{l+1})$
$(B_l,\, B_{l-1} \oplus D_1)$ $(l \ge 3)$	$-\alpha_1$	$2l - 3$	Z_{long}^{4l-5}	Yes	$\mathbb{Q}^{2l-1}$
$(C_l,\, A_{l-1} \oplus D_1)$ $(l \ge 2)$	$-\alpha_l$	$l - 1$	Z_{long}^{2l-1}	Yes	$\mathrm{LG}(l,\, \mathbb{C}^{2l})$
$(D_l,\, D_{l-1} \oplus D_1)$ $(l \ge 4)$	$-\alpha_1$	$2l - 4$	Z_{long}^{4l-7}	Yes	$\mathbb{Q}^{2l-2}$
$(D_l,\, A_{l-1} \oplus D_1)$ $(l \ge 4)$	one of $-\alpha_{l-1},\, -\alpha_l$	$2l - 4$	Z_{long}^{4l-7}	Yes	$\mathbb{S}_l$
$(G_2,\, A_2)$	$-\alpha_1$	2	Z_{short}^{7}		
$(F_4,\, A_2 \oplus A_2)$	$-\alpha_3$	4	Z_{long}^{15}		
$(E_6,\, A_2 \oplus A_2 \oplus A_2)$	$-\alpha_3$	6	Z_{long}^{21}		
$(E_6,\, D_5 \oplus D_1)$	one of $-\alpha_1,\, -\alpha_5$	10	Z_{long}^{21}	Yes	$\mathbb{OP}^2$
$(E_7,\, A_2 \oplus A_5)$	one of $-\alpha_3,\, -\alpha_5$	10	Z_{long}^{33}		
$(E_7,\, E_6 \oplus D_1)$	$-\alpha_1$	16	Z_{long}^{33}	Yes	E_7/P_1
$(E_8,\, E_6 \oplus A_2)$	$-\alpha_2$	18	Z_{long}^{57}		
$(E_8,\, A_8)$	$-\alpha_8$	20	Z_{long}^{57}		
$(E_8,\, A_4 \oplus A_4)$	$-\alpha_4,\, -(\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8)$	8	Z_{long}^{57}		

Table 4: Highest weight orbits for maximal proper reductive subalgebras $\mathfrak{l} < \mathfrak{s}$ of equal rank but not isotropy irreducible.

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