박 사 학 위 논 문
Ph.D. Dissertation

뤼나-뷔스트 이론의 관점으로 본 수반 다양체의 이차곡선 매개변수공간

Parameter Spaces of Conics in Adjoint Varieties and Luna-Vust Theory

2025

권 민 성 (權 玟 成 Kwon, Minseong)

한 국 과 학 기 술 원

Korea Advanced Institute of Science and Technology

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Parameter Spaces of Conics in Adjoint Varieties and Luna-Vust Theory

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초 록

본 학위논문에서는 수반 다양체 상의 이차곡선과, 그의 매개변수공간을 다룬다. 단순 리대수 $\mathfrak{g}$가 주어지면, 그에 대한 단일 연결 리군 G는 수반 다양체 $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$에 자연스럽게 작용하며, 이는 $Z_{\mathfrak{g}}$의 접촉 구조를 보존한다. 본 논문에서는 먼저 $Z_{\mathfrak{g}}$ 상의 트위스터 이차곡선, 즉 접촉 구조에 접하지 않는 이차곡선의 공간은 G가 추이적으로 작용하는 대칭 공간임을 증명한다. 이는 $Z_{\mathfrak{g}}$ 상의 이차곡선 공간이 구형임을 의미하므로, 이후에는 뤼나-뷔스트 이론을 통해 콤팩트화된 이차곡선 공간, 특히 힐베르트 스킴, 차우 스킴, 그리고 완비이차곡선 공간을 연구한다. 본 논문의 주요 결과는, 앞서 소개된 콤팩트화된 공간의 기하를 결정하는 조합론적 정보, 즉 뤼나-뷔스트 이론에서의 채색된 부채꼴을 계산한 것이다.

후반부에서는 주요 정리의 응용을 다룬다. 먼저 G의 작용에 대한 이차곡선의 궤도 및 켤레류를 분류하며, 특히 G_2-수반 다양체에서는 매끄러운 이차곡선이 접촉 구조에 접할 수 없음을 증명한다. 그후에는 힐베르트 스킴이 매끄러움을 증명하고, 마지막으로 힐베르트 스킴 상의 최소 유리 곡선을 $Z_{\mathfrak{g}}$ 상의 이차곡선을 이용해 묘사한다.

핵 심 낱 말 리군, 수반 다양체, 이차곡선, 뤼나-뷔스트 이론, 구형 다양체, 힐베르트 스킴

Abstract

For a simple Lie algebra $\mathfrak{g}$, we study spaces of conics on the adjoint variety $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$. First, we prove that for the simply connected Lie group G associated to $\mathfrak{g}$, the space of twistor conics, i.e. smooth conics transverse to the contact distribution D of $Z_{\mathfrak{g}}$, is a homogeneous G-symmetric variety. In particular, this implies that the space of smooth conics on $Z_{\mathfrak{g}}$ is a G-spherical variety. Then we apply Luna-Vust theory for spherical varieties to three compactifications of the space of smooth conics: the Hilbert scheme $\mathbf{H}_{\mathfrak{g}}$, the Chow scheme $\mathbf{C}_{\mathfrak{g}}$, and the space $\mathbf{CoC}_{\mathfrak{g}}$ of complete conics. Our main result is the computation of the colored fans of the three compactifications.

Next, we present several applications of the main result. Namely, the conjugacy classes of conics on $Z_{\mathfrak{g}}$ with respect to the natural G-action are explicitly described. Especially, we show that the G_2-adjoint variety does not contain a smooth conic tangent to the contact distribution. We also prove smoothness of the normalized Hilbert scheme $\mathbf{H}_{\mathfrak{g}}^{nor}$. Finally, we interpret minimal rational curves on $\mathbf{H}_{\mathfrak{g}}^{nor}$ in terms of conics on $Z_{\mathfrak{g}}$.

Keywords Lie group, adjoint variety, conic, Luna-Vust theory, spherical variety, Hilbert scheme

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Chapter 1. Introduction

We are working over $\mathbb{C}$, the field of complex numbers. Let $\mathfrak{g}$ be a simple Lie algebra, and G the associated simply connected Lie group. Then G naturally acts on $\mathfrak{g}$ via the adjoint representation. Moreover, in the projectivization $\mathbb{P}(\mathfrak{g})$, there exists a unique closed G-orbit $Z_\mathfrak{g}$, called the *adjoint variety* of $\mathfrak{g}$.

By its definition, the adjoint variety $Z_\mathfrak{g} \subset \mathbb{P}(\mathfrak{g})$ coincides with the projectivization of the minimal nonzero nilpotent orbit in $\mathfrak{g}$. It is well-known that a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is equipped with a G-invariant symplectic structure (the so-called *Kirillov-Kostant-Souriau structure*). Thus its projectivization $Z := \mathbb{P}\mathcal{O}$ admits a *contact distribution*, that is, a hyperplane subbundle D of the tangent bundle TZ such that the Lie bracket of vector fields induces a bundle morphism

$$\bigwedge^2 D \to TZ/D \quad \text{(quotient line bundle)}$$

which is everywhere non-degenerate.

In particular, the adjoint variety $Z_\mathfrak{g}$ can be viewed as a rational homogeneous space equipped with an invariant contact distribution. In fact, Boothby [4] shows that the adjoint varieties are the only rational homogeneous spaces admitting invariant contact structures. For this reason, the adjoint varieties are often called *homogeneous contact manifolds*. Moreover, the adjoint varieties are the only known examples of *Fano contact manifolds*, and it has been conjectured that every Fano contact manifold is isomorphic to one of the adjoint varieties (the so-called *LeBrun-Salamon conjecture*; see [2]).

On the other hand, since the adjoint variety $Z_\mathfrak{g}$ is a rational homogeneous space, it is covered a large family of rational curves, i.e. non-constant images of $\mathbb{P}^1$. Then it is natural to study a connection between the contact distribution of $Z_\mathfrak{g}$ and geometry of rational curves on $Z_\mathfrak{g}$. For example, it is known that if a rational curve on $Z_\mathfrak{g}$ is not tangent to the contact distribution, then its degree, with respect to the embedding into $\mathbb{P}(\mathfrak{g})$, must be at least 2.

Motivated by these facts, we study conics, i.e. rational curves of degree 2 on $Z_\mathfrak{g} \subset \mathbb{P}(\mathfrak{g})$, and their parameter spaces. Before stating our main theorems, we recall related results.

1.1 State of the Art

1.1.1 Rational Curves on Rational Homogeneous Spaces

As noted before, the adjoint variety $Z_\mathfrak{g}$ contains a lot of rational curves. Among them, geometry of lines, i.e. rational curves of degree 1, is well-understood. Namely, Hwang [15] proves that if $Z_\mathfrak{g} \ncong \mathbb{P}^N$ and $\ncong \mathbb{P}T^*\mathbb{P}^N$, then tangent directions of lines passing through a point $o \in Z_\mathfrak{g}$ form a homogeneous Legendrian submanifold of $\mathbb{P}(D_o)$ where D is the contact distribution of $Z_\mathfrak{g}$. Using this, Hwang shows that under the same assumption, $Z_\mathfrak{g}$ is rigid under Fano deformation.

More generally, Landsberg and Manivel [23] study linear subspaces on rational homogeneous spaces. Indeed, Landsberg and Manivel give a recipe to construct parameter spaces of linear subspaces on rational homogeneous spaces. (Such a parameter space is called a *Fano variety/scheme* in literature, while we do not use this terminology to avoid a confusion with a manifold whose anti-canonical bundle is ample.) In particular, the description of the space of lines on the adjoint variety $Z_\mathfrak{g}$ is recovered.

Beyond lines, geometry of rational curves of higher degree is more complicated. The main difficulty is that smooth rational curves of higher degree can degenerate to singular curves with several components. That is, the space of smooth rational curves is often non-compact, hence may allow various compactifications. For instance, one can compactify it using the *Chow scheme* (parametrizing algebraic cycles), the *Hilbert scheme* (parametrizing closed subschemes), the *Kontsevich moduli space* (parametrizing stable maps), and the *Simpson's moduli space of semi-stable sheaves* (see [10, §1] for the precise definitions). Furthermore, in the case of conics, one can consider the *space of complete conics* (parametrizing degenerations of pairs of a smooth conic and its dual conic), which is a classical object in enumerative geometry (see [42]).

The study on compactified spaces of rational curves is an active research area. An important result is that on a rational homogeneous space X, if we choose a homology class $\alpha \in H_2(X, \mathbb{Z})$, then the space of smooth rational curves representing α is irreducible, whenever it is nonempty. This statement is verified by several authors, including Kim and Pandharipande [18], Thomsen [40] and Perrin [34]. In particular, if X is a rational homogeneous space of Picard number 1, then the space of smooth rational curves of fixed degree on X is irreducible (possibly empty).

In the case where the degree of rational curves is at most 3, Chung, Kiem and Hong [10] construct explicit birational morphisms relating three compactifications introduced above: the Hilbert compactification, the Kontsevich compactification, and the Simpson compactification. Thus once we obtain a description of one of them, a description of the others follows. For example, for the space of conics, after taking the normalizations, the Hilbert compactification and the Kontsevich compactification can be constructed as blow-downs of a common smooth variety, where the blow-up loci are specified in [10, Theorem 3.7].

1.1.2 Family of Conics Parametrized by Riemannian Symmetric Spaces

After Boothby's characterization [4], Wolf [44] proves that the adjoint variety $Z_{\mathfrak{g}}$ admits a foliation by smooth conics, whose leaf space is a certain Riemannian symmetric space. In this subsection, we recall Wolf's theorem and its generalization. Though the results in this subsection would not be used in this article, they shall explain a connection between conics on $Z_{\mathfrak{g}}$ and the theory of spherical varieties (Subsection 1.1.3).

More precisely, Wolf [44] obtains a bijective correspondence between the adjoint varieties and *Wolf spaces*. Here, a *Wolf space* is a Riemannian symmetric space which is *quaternion-Kähler* (*QK* for short; see [3, Chapter 14] for the definition) and of positive curvature. Wolf spaces are real analytic manifolds, but not necessarily complex manifolds.

Wolf's correspondence [44] says that for each adjoint variety $Z_{\mathfrak{g}}$, there is a Wolf space $M_{\mathfrak{g}}$ equipped with a real analytic fibration $Z_{\mathfrak{g}} \to M_{\mathfrak{g}}$ whose fibers are smooth conics transverse to the contact distribution and have normal bundles $\simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\dim Z_{\mathfrak{g}}-1)}$. Conversely, every Wolf space arises in this way. (In fact, Wolf also obtains a similar correspondence between certain homogeneous domains in the adjoint varieties and QK symmetric spaces of negative curvature; see [44, Theorem 6.7].)

Salamon [38] generalizes Wolf's correspondence to arbitrary QK manifolds. For each QK manifold M, Salamon constructs a complex manifold Z equipped with a contact distribution D and a fibration $Z \to M$ generalizing Wolf's result: the fibers are smooth rational curves transverse to D and have normal bundles $\simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\dim Z-1)}$. Furthermore, Salamon's construction shows that there exists an anti-holomorphic involution $\theta : Z \to Z$ preserving D and each fiber of $Z \to M$. As a consequence, M, parametrizing a family of θ-invariant rational curves, can be considered as a totally real submanifold of the space of rational curves on Z with normal bundles $\simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\dim Z-1)}$. Nowadays, Z is called the *twistor space* of the QK manifold M. The fibers of $Z \to M$ are often called *twistor lines*, however in this article, to emphasize their TZ/D-degree (= 2), we call them *twistor conics*. For these results, we refer to [38] and [25, §1].

The other direction of Wolf's correspondence is generalized by LeBrun [25]. To state the result, we need two data: a complex contact manifold (Z, D) and a fixed-point-free anti-holomorphic involution $\theta : Z \to Z$ preserving D. Under this setting, LeBrun shows that if Y is the set of θ-invariant smooth rational curves $C \subset Z$ transverse to D and with normal bundles $\simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(\dim Z-1)}$, then Y is a QK manifold (possibly pseudo-Riemannian; see [25, Theorem 1.3]). Using this result, LeBrun [25, §2] constructs new examples of QK manifolds.

Finally, based on LeBrun's theorem, Dufour [12] provides a recipe of QK manifolds, starting from parabolic geometries modeled on the G_2-adjoint variety Z_{G_2}, and then describes the resulting QK metrics explicitly. In the proof, the study on deformations of double lines (which are singular conics) on the G_2-adjoint variety Z_{G_2} plays an important role. See [12, Théorèmes 108, 126] for details.

1.1.3 Luna-Vust Theory for Spherical Varieties

As explained in Subsection 1.1.2, the adjoint variety $Z_{\mathfrak{g}}$ admits a real analytic family of smooth conics parameterized by a Riemannian symmetric space $M_{\mathfrak{g}}$. Furthermore, $M_{\mathfrak{g}}$ can be viewed as a totally real submanifold of the space of smooth conics on $Z_{\mathfrak{g}}$. From these facts, one can expect that the space of smooth conics may contain the *complexification* of $M_{\mathfrak{g}}$, i.e. a homogeneous space $(G_{\mathbb{R}})^{\mathbb{C}}/(K_{\mathbb{R}})^{\mathbb{C}}$ where $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$ are compact real Lie groups such that $M_{\mathfrak{g}} \simeq G_{\mathbb{R}}/K_{\mathbb{R}}$. This observation leads us to the theory of *spherical varieties*, since the homogeneous space $(G_{\mathbb{R}})^{\mathbb{C}}/(K_{\mathbb{R}})^{\mathbb{C}}$ is a *symmetric variety* (Definition 2.3.9), hence in particular a *spherical variety*.

Let us briefly introduce the notion of spherical varieties and their embedding theory. For a connected reductive group G', a normal G'-variety is called *(G'-)spherical* if a Borel subgroup of G' has an open orbit. The class of spherical varieties includes a lot of classical examples of almost homogeneous varieties: toric varieties, rational homogeneous spaces, symmetric varieties, etc. Observe that by the definition, a G'-spherical variety X contains an open G'-orbit, say O. In this case, we say that X is an *O-embedding*.

Recently, there has been a huge progress in the classification of spherical varieties. The program consists of two steps: (1) given a homogeneous spherical variety O, classify all O-embeddings, and (2) classify homogeneous spherical varieties. The first step (1) is completed by Luna and Vust [27], while the second step (2) is achieved more recently, contributed by several researchers (including Bravi, Cupit-Foutou, Losev, Luna and Pezzini; we refer to [41, §30.11–12] and the references therein). In this article, we are mainly interested in the classification of a fixed homogeneous spherical variety, so let us summarize the result of Luna and Vust on the first step (1).

In [27], Luna and Vust show that given a homogeneous spherical variety O, there is a bijective correspondence between O-embeddings and certain combinatorial objects, called *colored fans* (modulo isomorphisms). A colored fan is a finite collection of pairs of a polyhedral cone and a finite set (see Definition 2.3.2), and so O-embeddings are classified in terms of finite combinatorial data. For example, if G' is a torus and $G' = O$, then O-embeddings are exactly toric varieties, and the colored fan of a toric variety can be identified with the associated fan, which appears in the standard theory of toric varieties ([31]). In fact, roughly, one can say that the colored fan of a spherical variety plays a role of the fan of a toric variety. For example, for spherical varieties, we have an orbit-cone correspondence (Lemma 2.3.4), and a smoothness criterion ([13]) in terms of the colored fans. More details on Luna-Vust theory can be found in Section 2.3.

1.2 Main Results and Structure of the Article

Now we explain the content of this article. The precise statements for the main theorems are given in Section 3.2, and in this section, we present them only in a simplified form. Recall that $\mathfrak{g}$ is a simple Lie algebra, and G is the associated simply connected Lie group. We are mainly interested in the irreducible component of the space of smooth conics on $Z_{\mathfrak{g}}$, parametrizing twistor conics; denote it by $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$. In fact, if $\mathfrak{g}$ is not of type A, then $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$ coincides with the whole space of smooth conics, see Section 3.1. For the other components in the case where $\mathfrak{g}$ is of type A, the situation becomes simpler, and we discuss them in Subsection 3.1.2.

In Chapter 2, we review known results which are necessary in our study: the contact distribution on the adjoint variety $Z_{\mathfrak{g}}$ (Section 2.1), the construction of the spaces of smooth rational curves on rational homogeneous spaces (Section 2.2), and Luna-Vust theory for symmetric varieties (Section 2.3). Namely, we introduce three compactifications of $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$: the (semi-normalized) Hilbert scheme $\mathbf{H}_{\mathfrak{g}}$, the Chow scheme $\mathbf{C}_{\mathfrak{g}}$, and the space $\mathbf{CoC}_{\mathfrak{g}}$ of complete conics. These are projective compactifications of $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$, and their normalizations are related via G-equivariant birational morphisms

$$\mathbf{CoC}_{\mathfrak{g}}^{nor} \to \mathbf{H}_{\mathfrak{g}}^{nor} \to \mathbf{C}_{\mathfrak{g}}^{nor}.$$

In Chapter 3, we study smooth conics on $Z_{\mathfrak{g}}$ and their deformations. The following are our first main theorem:

Main Theorem 1 (Theorem 3.2.1)**.** *Twistor conics on $Z_{\mathfrak{g}}$, i.e. smooth conics transverse to the contact structure, form an open G-orbit $O_{\mathfrak{g}}$ in $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$, which is isomorphic to a homogeneous G-symmetric variety.*

Its proof is given in Section 3.3. In Chapter 3, we also study smooth conics passing through a given point, and prove the following theorem:

Main Theorem 2 (Theorem 3.3.4, Corollary 3.4.2)**.** *Let $o \in Z_{\mathfrak{g}}$ be a point, $v \in T_o Z_{\mathfrak{g}}$ a nonzero tangent vector, and $D_o \subset T_o Z_{\mathfrak{g}}$ the contact hyperplane at o.*

1. *If $v \notin D_o$, then there exists a unique twistor conic tangent to v. That is, the space of twistor conics passing through o is identified with $\mathbb{P}(T_o Z_{\mathfrak{g}}) \smallsetminus \mathbb{P}(D_o) \simeq \mathbb{C}^{\dim Z_{\mathfrak{g}} - 1}$.*
2. *If $\mathfrak{g}$ is not of type C and $[v]$ is a general element of $\mathbb{P}(D_o)$, then there is no smooth conic tangent to v.*

(If $\mathfrak{g}$ is of type C, then there is a smooth conic in every direction; see Subsection 3.1.1.) The first statement is proven in Section 3.3, and the second is proven in Section 3.4. Here, the meaning of *general element of* $\mathbb{P}(D_o)$ in the last statement is specified in Proposition 3.4.1. In the last part of this chapter (Section 3.5), we classify B-fixed points in the compactifications $\mathbf{C}_{\mathfrak{g}}$, $\mathbf{H}_{\mathfrak{g}}$ and $\mathbf{CoC}_{\mathfrak{g}}$ for a Borel subgroup B of G, which represent the 'most singular' deformations of smooth conics on $Z_{\mathfrak{g}}$.

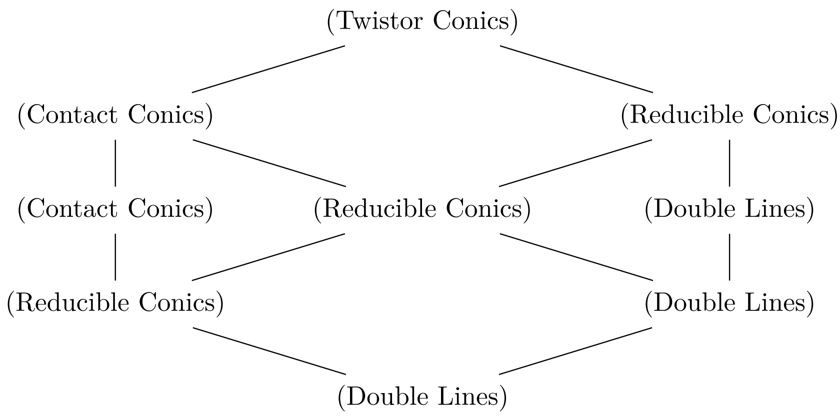
Chapter 4 is the core of this article. From this chapter, we regard the compactifications $\mathbf{C}_{\mathfrak{g}}^{nor}$, $\mathbf{H}_{\mathfrak{g}}^{nor}$ and $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ as spherical varieties, and then compute their colored fans as $O_{\mathfrak{g}}$-embeddings:

Main Theorem 3 (Theorem 3.2.2). *The normalizations* $\mathbf{C}_{\mathfrak{g}}^{nor}$, $\mathbf{H}_{\mathfrak{g}}^{nor}$ *and* $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ *are* G*-symmetric varieties, and their colored fans are given in Tables 3.2–3.4.*

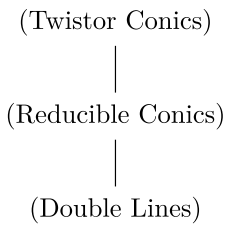
Chapter 4 is devoted to the proof of the theorem. The main ingredients of the proof are (1) Luna-Vust theory for symmetric varieties (Sections 2.3, 3.2), and (2) the classification of B-fixed points (Section 3.5). Using these, we compute the colored fan of each compactification case by case.

In the final Chapter 5, we introduce several applications of the main theorems. Let us introduce some consequences:

- A classification of G-conjugacy classes of conics on $Z_{\mathfrak{g}}$ (Section 5.2). For example, when $\mathfrak{g}$ is of exceptional type, we show that G-conjugacy classes of conics can be visualized as the following diagrams:



  if $\mathfrak{g}$ is E_r ($r = 6, 7, 8$) or F_4, and



  if $\mathfrak{g} = G_2$. Here, a *contact* conic means a smooth conic tangent to the contact distribution D (Section 3.1), and we draw an edge whenever conics in the upper class can degenerate to conics in the lower class (for the precise definition and similar diagrams for other $\mathfrak{g}$, see Figures 5.1–5.8 and the discussion after Theorem 5.2.4). In particular, we conclude that in the G_2-adjoint variety Z_{G_2}, every smooth conic is transverse to the contact distribution.

- Smoothness of the normalized Hilbert scheme $\mathbf{H}_{\mathfrak{g}}^{nor}$ (Corollary 5.3.1). This result also follows from [10, Proposition 3.6], and we shall give a different proof using spherical geometry.

- A description of the *variety of minimal rational tangents* (*VMRT* for short; Definition 5.4.1) of $\mathbf{H}_{\mathfrak{g}}^{nor}$ in terms of lines and conics on $Z_{\mathfrak{g}}$ (Section 5.4 and Figure 5.9).

Remark 1.2.1. A part of this article has appeared in the author's preprint [22]. There are several changes in this article, and the main differences are as follows:

- The application to VMRT of $\mathbf{H}_{\mathfrak{g}}^{nor}$ is new. The author would like to thank his Ph.D. advisor Jun-Muk Hwang for sharing the observation in Remark 5.4.8, and for suggesting finding its geometric interpretation.
- This article includes the study on $\mathbf{CoC}_{\mathfrak{g}}$, the space of complete conics, while it is not treated in [22]. The author is very grateful to Michel Brion for the suggestion to study the space of complete conics and for helpful discussions.
- This article covers every simple Lie algebra $\mathfrak{g}$, while in [22], $\mathfrak{g}$ is assumed to be not of type A or C. The author would like to thank Nicolas Perrin for pointing out his misunderstanding in the case of type A. The author is also grateful to DongSeon Hwang for encouraging him to consider all the cases.
- The presentation of the proofs of the main theorems is improved. Especially, the computation of the colored fans in the case where $\mathfrak{g} = B_3$ is simplified. The author would like to thank Jaehyun Hong for valuable comments and discussions.

Chapter 2. Preliminaries

In this chapter, we explain our notation and review several known facts. First of all, our base field is $\mathbb{C}$, the field of complex numbers, and every *scheme* is assumed to be locally of finite type over $\mathbb{C}$. A *variety* means an integral separated scheme of finite type over $\mathbb{C}$, and a *point* in a scheme means a closed point. For a vector space V, $\mathbb{P}(V) := V - \{0\}/\mathbb{C}^\times$ denotes the space of 1-dimensional subspaces of V.

2.1 Adjoint Variety and Contact Distribution

Our main reference on Lie theory is [32]. Let $\mathfrak{g}$ be a semi-simple Lie algebra (until we define the *adjoint variety*). Let G be the simply connected Lie group associated to the Lie algebra $\mathfrak{g}$. Choose a maximal torus T in G, and a Borel subgroup B containing T. We denote the Lie algebras of T and B by $\mathfrak{t}$ and $\mathfrak{b}$, respectively. The set of roots and the set of simple roots are denoted by R and S, respectively. When $\mathfrak{g}$ is simple, we use the numbering $S = \{\alpha_1, \ldots, \alpha_{\operatorname{rank}\mathfrak{g}}\}$ of simple roots given in [32, Reference Chapter, Table 1] (which is different from the one in [6], especially for exceptional Lie algebras other than G_2). For each root $\alpha \in R$, $\mathfrak{g}_\alpha$ means the root space corresponding to α so that the root decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

The character group of T is denoted by $\chi(T)$, and each character $\lambda \in \chi(T)$ is regarded as a linear functional on $\mathfrak{t}$. In this notation, if $H \in \mathfrak{t}$, then the value of the character corresponding to λ at $\exp(H)$ is equal to $e^{\lambda(H)}$. The bracket $\langle\,,\,\rangle$ means the Killing form on $\mathfrak{g}$, and the dual of a root $\alpha \in R$ is denoted by $H_\alpha \in \mathfrak{t}$. More precisely, H_α is the element of $\mathfrak{t}$ satisfying $\langle H_\alpha, H\rangle = \alpha(H)$ for all $H \in \mathfrak{t}$. The pairing of two roots α and β is defined as $\langle \alpha, \beta\rangle := \langle H_\alpha, H_\beta\rangle$, which extends to an inner product on $\chi(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The natural pairing of $\chi(T)$ and its dual $\chi_*(T)$ is also denoted by $\langle \lambda, \mu\rangle$ for $\lambda \in \chi(T)$ and $\mu \in \chi_*(T)$ so that the Cartan integer is given by $\langle \alpha | \beta\rangle = \langle \alpha, \beta^\vee\rangle$ for $\alpha, \beta \in R$ where $\beta^\vee$ is the coroot corresponding to β. A nonzero vector in $\mathfrak{g}_\alpha$ for some $\alpha \in R$ is called a root vector, which is often denoted by E_α. If a collection $\{E_\alpha \in \mathfrak{g}_\alpha : \alpha \in R\}$ of root vectors is given, we define $N_{\alpha,\beta}$ for $\alpha, \beta \in R$ to be the complex number satisfying

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} \cdot E_{\alpha+\beta}$$

if $\alpha + \beta \in R$, and $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin R$.

For a nonempty subset $I \subset S$, we denote by P_I the parabolic subgroup containing B generated by the complement $S \setminus I$ of I. That is, the Lie algebra of P_I is

$$\mathfrak{p}_I := \mathfrak{b} \oplus \bigoplus_{\alpha \in R^+ \cap \operatorname{span}(S \setminus I)} \mathfrak{g}_{-\alpha}$$

where R^+ is the set of all positive roots. The opposite parabolic subgroup of P_I is denoted by P_I^-. We define $W = W_G$ to be the Weyl group of (G, T), and W_{G,P_I} means the subgroup of W generated by reflections with respect to $\alpha \in S \setminus I$ so that $P_I = B \cdot W_{G,P_I} \cdot B$.

From now on, assume that $\mathfrak{g}$ be a simple Lie algebra. Then the adjoint representation is irreducible, and its highest weight is the highest root of $\mathfrak{g}$, denoted by $\rho \in R$ (with respect to the Borel subgroup B). Thus $\mathbb{P}(\mathfrak{g})$ contains the unique closed G-orbit, which is the G-orbit of long root vectors. This projective subvariety of $\mathbb{P}(\mathfrak{g})$ is called the *adjoint variety* and denoted by $Z_\mathfrak{g}$. By its construction, $Z_\mathfrak{g}$ is isomorphic

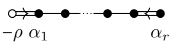
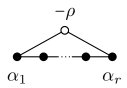
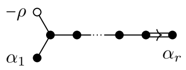
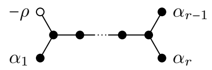
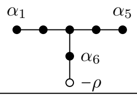
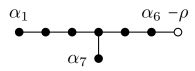
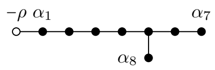
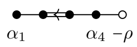
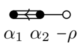
$\mathfrak{g}$	Extended Dynkin diagram of $\mathfrak{g}$	$N(\rho)$	Classical description of $Z_{\mathfrak{g}}$	$\dim Z_{\mathfrak{g}}$	n
C_r $(r \geq 1)$		$\{\alpha_1\}$	$\mathbb{P}^{2r-1}$	$2r-1$	$r-1$
A_r $(r \geq 2)$		$\{\alpha_1, \alpha_r\}$	$\mathrm{Fl}_{1,r}(\mathbb{C}^{r+1})$	$2r-1$	$r-1$
B_r $(r \geq 3)$		$\{\alpha_2\}$	$\mathrm{OG}(2, \mathbb{C}^{2r+1})$	$4r-5$	$2r-3$
D_r $(r \geq 4)$		$\{\alpha_2\}$	$\mathrm{OG}(2, \mathbb{C}^{2r})$	$4r-7$	$2r-4$
E_6		$\{\alpha_6\}$	-	21	10
E_7		$\{\alpha_6\}$	-	33	16
E_8		$\{\alpha_1\}$	-	57	28
F_4		$\{\alpha_4\}$	-	15	7
G_2		$\{\alpha_2\}$	-	5	2

Table 2.1: Information on $Z_{\mathfrak{g}}$.

to a rational homogeneous space G/P where $o := [\mathfrak{g}_\rho] \in \mathbb{P}(\mathfrak{g})$ and P is the *isotropy group* of o in G, i.e. $P := \mathrm{Stab}_G(o)$. The isotropy group P is a parabolic subgroup containing B and its Lie algebra is

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R, \langle \alpha, \rho \rangle \geq 0} \mathfrak{g}_\alpha = \mathfrak{p}_{N(\rho)}$$

where $N(\rho) := \{\alpha_i \in S : \langle \alpha_i, \rho \rangle \neq 0\}$. In other words, $N(\rho)$ consists of simple roots which are neighbors of $-\rho$ in the extended Dynkin diagram of $\mathfrak{g}$. Using this description, one can easily show that $\dim(Z_{\mathfrak{g}}) = 2n+1$ for some $n \in \mathbb{Z}_{\geq 0}$. See Table 2.1 for the extended Dynkin diagram of $Z_{\mathfrak{g}}$ and the value of n. In the same table, we provide a description of $Z_{\mathfrak{g}}$ in the case where $\mathfrak{g}$ is of classical type. Here, $\mathrm{Fl}_{1,r}(\mathbb{C}^{r+1})$ is the partial flag variety (parametrizing pairs of lines and hyperplanes in $\mathbb{C}^{r+1}$), and $\mathrm{OG}(2, \mathbb{C}^N)$ is the orthogonal Grassmannian (parametrizing isotropic 2-planes in $\mathbb{C}^N$).

The adjoint variety $Z_{\mathfrak{g}}$ comes with a hyperplane distribution described as follows. Consider a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where

$$\mathfrak{g}_0 := \mathfrak{t} \oplus \bigoplus_{\alpha \in R : \langle \alpha, \rho \rangle = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{\pm 1} := \bigoplus_{\alpha \in R^{\pm} \setminus \{\pm\rho\} : \langle \alpha, \rho \rangle \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{\pm 2} := \mathfrak{g}_{\pm\rho}.$$

It makes $\mathfrak{g}$ a graded Lie algebra, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ where $\mathfrak{g}_k := 0$ for all $k \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$. Let us write $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_i$. Then $\mathfrak{p} = \mathfrak{g}^0$, and each $\mathfrak{g}^i$ is a P-module. In the tangent space $T_o Z_{\mathfrak{g}}$ of $Z_{\mathfrak{g}}$ at o, which is identified with $\mathfrak{g}/\mathfrak{p}$ as a P-module, consider a hyperplane D_o identified with $\mathfrak{g}^{-1}/\mathfrak{p} (\subset \mathfrak{g}/\mathfrak{p})$. Since D_o is P-invariant, the G-action on $Z_{\mathfrak{g}}$ induces a well-defined G-invariant vector subbundle $D \simeq G \times_P D_o$ of $TZ_{\mathfrak{g}} \simeq G \times_P (\mathfrak{g}/\mathfrak{p})$ extending D_o. This hyperplane distribution D is called the *contact distribution* on the adjoint variety.

Remark 2.1.1. There is a notion of contact distribution on a complex manifold, and we refer to [24]. By the result of Boothby [4], the adjoint varieties can be characterized as rational homogeneous spaces equipped with invariant contact distributions.

The quotient line bundle $TZ_{\mathfrak{g}}/D$, called the *contact line bundle*, can be described as follows. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(-1)$ on $\mathbb{P}(\mathfrak{g})$. Its restriction on $Z_{\mathfrak{g}}$ is a G-homogeneous line bundle, isomorphic to $G \times_P \mathfrak{g}_\rho$. Observe that the Killing form of $\mathfrak{g}$ identifies $\mathfrak{g}/\mathfrak{g}^{-1}$ and the dual of $\mathfrak{g}_2$ as P-modules. Therefore we have

$$\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}} \simeq G \times_P (\mathfrak{g}/\mathfrak{g}^{-1}) \simeq TZ_{\mathfrak{g}}/D.$$

Next, we introduce another description of the gradation $\mathfrak{g} = \bigoplus_{m=-2}^{2} \mathfrak{g}_m$. For a simple root $\alpha_i \in S$ and a root $\alpha \in R$, let $m_i(\alpha)$ be the coefficient of α_i in α.

- If $\mathfrak{g}$ is not of type A, then there is exactly one simple root, say α_{j_0}, which is not orthogonal to the highest root ρ (Table 2.1). In this case,

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in R : m_{j_0}(\alpha) = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_m = \bigoplus_{\alpha \in R : m_{j_0}(\alpha) = m} \mathfrak{g}_m, \quad \forall m \neq 0.$$

  Moreover, we have

$$2 = \langle \rho \,|\, \rho \rangle = m_{j_0}(\rho) \cdot \langle \alpha_{j_0} \,|\, \rho \rangle = 2 \cdot \langle \alpha_{j_0} \,|\, \rho \rangle,$$

  hence for $\alpha \in R$,

$$\langle \alpha \,|\, \rho \rangle = m_{j_0}(\alpha) \cdot \langle \alpha_{j_0} \,|\, \rho \rangle = m_{j_0}(\alpha).$$

  Note that α_{j_0} is a long root if and only if $\mathfrak{g}$ is not of type C.

- If $\mathfrak{g} = A_r$ $(r \geq 2)$, then $N(\rho) = \{\alpha_1, \alpha_r\}$, and

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in R : m_1(\alpha) + m_r(\alpha) = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_m = \bigoplus_{\alpha \in R : m_1(\alpha) + m_r(\alpha) = m} \mathfrak{g}_m, \quad \forall m \neq 0.$$

  As $\langle \alpha_1 \,|\, \rho \rangle = \langle \alpha_r \,|\, \rho \rangle = 1$, for $\alpha \in R$, we have

$$\langle \alpha \,|\, \rho \rangle = m_1(\alpha) + m_r(\alpha).$$

To summarize, for arbitrary $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in R : \langle \alpha \,|\, \rho \rangle = 0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_m = \bigoplus_{\alpha \in R : \langle \alpha \,|\, \rho \rangle = m} \mathfrak{g}_m, \quad \forall m \neq 0.$$

2.2 Parameter Spaces of Rational Curves

Our main goal is to understand spaces of conics on the adjoint varieties. For this purpose, we recall several parameter spaces of subobjects of a given projective variety.

2.2.1 Hilbert Schemes and Chow Schemes

Let us recall the relation between the Hilbert scheme and the Chow scheme. Our main reference is [20, Ch. I]. Consider a projective variety X equipped with an ample line bundle $\mathcal{L}$. For each polynomial $p(m) \in \mathbb{Q}[m]$, there is a projective scheme $\mathrm{Hilb}_{p(m)}(X, \mathcal{L})$, called the *Hilbert scheme*, which is the moduli space of closed subschemes of X with Hilbert polynomial $p(m)$ with respect to $\mathcal{L}$. More precisely, following [20, Theorem I.1.4], $\mathrm{Hilb}_{p(m)}(X, \mathcal{L})$ is defined to be the scheme representing the functor

$$\begin{array}{ccc} \{\text{schemes}\} & \to & \{\text{sets}\} \\ T & \mapsto & \left\{ \begin{array}{c} \text{closed subschemes of } X \times T \text{ which are flat, proper over } T \\ \text{and whose Hilbert polynomials over } T \text{ with respect to } \mathcal{L} \text{ are } p(m) \end{array} \right\}. \end{array}$$

We also write $\mathrm{Hilb}(X) := \bigsqcup_{p(m) \in \mathbb{Q}[m]} \mathrm{Hilb}_{p(m)}(X, \mathcal{L})$ (disjoint union), and call it the *Hilbert scheme*. In particular, points of $\mathrm{Hilb}(X)$ ($\mathrm{Hilb}_{p(m)}(X, \mathcal{L})$, respectively) correspond to closed subschemes of X (which have Hilbert polynomial $p(m)$ with respect to $\mathcal{L}$, respectively). One advantage of using the Hilbert scheme is that its infinitesimal structure is well-understood. For example:

Theorem 2.2.1 ([20, Theorem I.2.8 and Proposition I.2.14])**.** *Let X be a projective variety, and $V \subset X$ a closed subscheme with its ideal sheaf $\mathcal{I}_V$. Then there is a natural isomorphism*

$$T_{[V]} Hilb(X) \simeq Hom_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$$

where the left hand side means the Zariski tangent space at the point $[V]$. If furthermore X is smooth, V is a local complete intersection, and $H^1(V, N_{V/X}) = 0$ where $N_{V/X}$ is the dual of the conormal sheaf $\mathcal{I}_V/\mathcal{I}_V^2$ (which is locally free), then $Hilb(X)$ is smooth at $[V]$ and its tangent space is given by

$$T_{[V]} Hilb(X) \simeq H^0(V, N_{V/X}).$$

On the other hand, for $d \in \mathbb{Z}_{\geq 0}$ and $d' \in \mathbb{Z}_{>0}$, there is another projective scheme $\mathrm{Chow}_{d,d'}(X, \mathcal{L})$, called the *Chow scheme*, which is the moduli space of non-negative proper algebraic d-cycles of $\mathcal{L}$-degree d' in X. As before, following [20, Theorem I.3.21], $\mathrm{Chow}_{d,d'}(X, \mathcal{L})$ is defined to be the scheme representing the functor

$$\begin{array}{ccc} \{\text{semi-normal schemes}\} & \to & \{\text{sets}\} \\ T & \mapsto & \left\{ \begin{array}{c} \text{well-defined families of non-negative, proper, algebraic cycles} \\ \text{of dimension } d \text{ and degree } d' \text{ of } X \times T/T \\ \text{(see [20, Definition I.3.10])} \end{array} \right\}. \end{array}$$

Here, semi-normality is defined as follows:

Definition 2.2.2 ([20, §I.7.2], [21, §10.2])**.** Let X be a reduced scheme.

1. The *semi-normalization* of X is a scheme X^{sn} satisfying the following conditions:
   - (a) X^{sn} is a reduced scheme equipped with a bijective morphism $X^{sn} \to X$;
   - (b) The normalization $X^{nor} \to X$ factors through the morphism $X^{sn} \to X$, i.e. $X^{nor} \to X^{sn} \to X$; and
   - (c) If Y is a scheme satisfying the conditions (a) and (b), then there exists a unique factorization $X^{sn} \to Y \to X$.

2. X is called *semi-normal* if it is isomorphic to its semi-normalization X^{sn}.

There is a natural morphism from the semi-normalized Hilbert scheme to the Chow scheme, which sends a subscheme to its fundamental cycle. Here, for a closed subscheme $V \subset X$ of dimension d, the *fundamental cycle* of V is an effective d-cycle on X defined as

$$FC(V) := \sum_{v_i : \dim V_i = d} \mathrm{length}(\mathcal{O}_{v_i, V}) \cdot [V_i]$$

where v_i runs over the generic points of V and V_i is the corresponding irreducible component of V^{red} ([20, Definition I.3.1.3]).

Theorem 2.2.3 ([20, Theorems I.6.6, I.7.3.1])**.** *For a projective variety X with an ample line bundle $\mathcal{L}$ and a polynomial $p(m) \in \mathbb{Q}[m]$ of degree d, FC induces a morphism*

$$FC : Hilb^{sn}_{p(m)}(X, \mathcal{L}) \to \coprod_{d'=1}^{\infty} Chow_{d,d'}(X, \mathcal{L}), \quad [V] \mapsto FC(V)$$

satisfying the following condition: For a closed subscheme $V \subset X$ with Hilbert polynomial $p(m)$ with respect to $\mathcal{L}$, FC is a local isomorphism near the point $[V] \in Hilb^{sn}_{p(m)}(X, \mathcal{L})$ if V is reduced, has pure dimension and satisfies Serre's condition S_2. Here, $Hilb^{sn}_{p(m)}(X, \mathcal{L})$ is the semi-normalization of the reduced scheme $(Hilb_{p(m)}(X, \mathcal{L}))^{red}$.

Note that since the semi-normalization morphism is bijective, we may identify points in the Hilbert scheme and points in its semi-normalization.

2.2.2 Spaces of Smooth Rational Curves

From now on, we focus on parameter spaces of rational curves. Suppose that X is a projective variety, equipped with an ample line bundle $\mathcal{L}$. A *rational curve* on X means the image of a non-constant morphism $\mathbb{P}^1 \to X$. Then for each $d \in \mathbb{Z}_{>0}$, there exists a quasi-projective variety parametrizing rational curves of $\mathcal{L}$-degree d on X, denoted by $\mathrm{RatCurves}_d(X, \mathcal{L})$. Roughly, $\mathrm{RatCurves}_d(X, \mathcal{L})$ is defined as the normalization of the locus of the fundamental cycles of rational curves in $\mathrm{Chow}_{1,d}(X, \mathcal{L})$. See [20, Definition–Proposition II.2.11] for details. Alternatively, $\mathrm{RatCurves}(X) := \coprod_{d \geq 1} \mathrm{RatCurves}_d(X, \mathcal{L})$ can be constructed as the $\mathrm{Aut}(\mathbb{P}^1)(= PGL_2)$-quotient of $\mathrm{Hom}^{nor}_{bir}(\mathbb{P}^1, X)$, the normalization of the space $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ of morphisms $\mathbb{P}^1 \to X$ which are birational onto their images ([20, Theorem II.2.15]).

Now assume that X is smooth, and consider a non-constant morphism $f : \mathbb{P}^1 \to X$. Following [20, Definition II.3.1], for a closed subscheme $B \subset \mathbb{P}^1$ and its ideal $\mathcal{I}_B \subset \mathcal{O}_{\mathbb{P}^1}$, we say that f is *free over* B if $H^1(\mathbb{P}^1, f^*TX \otimes \mathcal{I}_B) = 0$ and $f^*TX \otimes \mathcal{I}_B$ is generated by global sections. Recall that by Grothendieck's theorem, f^*TX is split into the direct sum of line bundles, say $f^*TX \simeq \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some integers $a_1 \geq \cdots \geq a_{\dim X}$. Thus if $B = \varnothing$ (i.e. $\mathcal{I}_B = \mathcal{O}_{\mathbb{P}^1}$), then f is free over B if and only if $a_{\dim X} \geq 0$. In this case, we simply say that f is *free*. On the other hand, if B is a point (so that $\mathcal{I}_B \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$), then f is free over B if and only if $a_{\dim X} \geq 1$.

Theorem 2.2.4 ([20, Theorem II.3.11])**.** *Let X be a smooth projective variety, and $B \subset \mathbb{P}^1$ a finite subscheme of length ≤ 2, possibly empty. For a morphism $g : B \to X$, there exist countably many closed subvarieties $V_i(B, g) \subsetneq X$ such that if $f : \mathbb{P}^1 \to X$ is a non-constant morphism satisfying $f|_B = g$ and $im(f) \not\subset \cup_i V_i(B, g)$, then f is free over g.*

In other words, one can say that a non-constant morphism $\mathbb{P}^1 \to X$ whose image passes through a *very general point* is free. In particular, if X is homogeneous under the action of $\mathrm{Aut}(X)$, then every non-constant morphism $\mathbb{P}^1 \to X$ is free.

Corollary 2.2.5. *Assume that X is a projective variety which is homogeneous under the action of Aut(X). Let Hilb(X) be the Hilbert scheme of X, and $\mathbf{R}(X) \subset$ Hilb(X) be the locus of smooth rational curves on X. Then $\mathbf{R}(X)$ is an open subscheme of Hilb(X), and it is smooth.*

Proof. $\mathbf{R}(X)$ is an open subscheme of $\mathrm{Hilb}(X)$ since being smooth is an open condition in a proper and flat family. Thus it is enough to show that if $C \subset X$ is a smooth rational curve, then $\mathrm{Hilb}(X)$ is smooth at the point $[C]$. In fact, since X is homogeneous, the embedding $\mathbb{P}^1 \xrightarrow{\simeq} C \subset X$ is a free morphism by Theorem 2.2.4. It means that $TX|_C$ is globally generated, hence the normal bundle $N_{C/X}$ is isomorphic to $\bigoplus_{i=1}^{\dim X} \mathcal{O}_C(a_i)$ for some integers $a_i \geq 0$. In particular, $H^1(C, N_{C/X}) = 0$, and so $[C]$ is a smooth point by Theorem 2.2.1. □

From now on, assume that X is a rational homogeneous space, i.e. X is homogeneous under an action of a reductive group. By Corollary 2.2.5, the space $\mathbf{R}(X)$ of smooth rational curves is smooth. By its construction, $\mathbf{R}(X) = \coprod_{p(m)\in\mathbb{Q}[m]} \mathbf{R}_{p(m)}(X, \mathcal{L})$ where $\mathcal{L}$ is an ample line bundle and $\mathbf{R}_{p(m)}(X, \mathcal{L}) := \mathbf{R}(X) \cap \mathrm{Hilb}_{p(m)}(X, \mathcal{L})$. Recall that the Picard group $\mathrm{Pic}(X)$ is isomorphic to $H^2(X, \mathbb{Z})$, and the (co)homology groups $H^2(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ are lattices of finite rank which are dual to each other. In other words, a homology class $\check{\alpha} \in H_2(X, \mathbb{Z})$ is uniquely determined by $\deg_{\mathcal{M}}(\check{\alpha})$, $\forall \mathcal{M} \in \mathrm{Pic}(X)$. Thus any two members of $\mathbf{R}_{p(m)}(X, \mathcal{L})$ represent the same homology class, hence given $\check{\alpha} \in H_2(X, \mathbb{Z})$,

$$\mathbf{R}_{\check{\alpha}}(X) := \{[C] \in \mathbf{R}(X) : C \text{ represents } \check{\alpha} \text{ in } H_2(X, \mathbb{Z})\}$$

is a well-defined open subscheme of $\mathbf{R}(X)$.

Theorem 2.2.6 ([18], [40], [34])**.** *For a rational homogeneous space X and a homology class $\check{\alpha} \in H_2(X, \mathbb{Z})$, $\mathbf{R}_{\check{\alpha}}(X)$ is irreducible whenever it is nonempty. That is, $\mathbf{R}_{\check{\alpha}}(X)$ is a smooth quasi-projective variety.*

Finally, we review the description of spaces of lines on rational homogeneous spaces, following [23, Section 4]. Let $X = G/P_I$ be a rational homogeneous space, G being a simply connected simple Lie group and nonempty $I \subset S$. Then $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ is isomorphic to the sublattice of the weight lattice, generated by the fundamental weights corresponding to elements of I. Indeed, if $\alpha_i \in I$ and ω_i is the corresponding fundamental weight, then the line bundle associated to ω_i can be obtained as the pull-back of $\mathcal{O}_{\mathbb{P}(V_i)}(1)$ via the morphism

$$X = G/P_I \to G/P_{\alpha_i} \subset \mathbb{P}(V_i)$$

where V_i is the fundamental representation associated to ω_i. Therefore the line bundle $\mathcal{L}_I$ defined by $\sum_{\alpha_i\in I} \omega_i$ can be obtained as the pull-back of $\mathcal{O}_{\mathbb{P}(\bigotimes_{\alpha_i\in I} V_i)}(1)$ via

$$\varphi_I : X = G/P_I \to \prod_{\alpha_i\in I} G/P_{\alpha_i} \hookrightarrow \prod_{\alpha_i\in I} \mathbb{P}(V_i) \hookrightarrow \mathbb{P}\left(\bigotimes_{\alpha_i\in I} V_i\right).$$

In fact, φ_I is an embedding, i.e. $\mathcal{L}_I$ is very ample. If $C \subset X$ is a $\mathcal{L}_I$-line, i.e. a rational curve of $\mathcal{L}_I$-degree 1, then there exists $\alpha_i \in I$ such that the homology class of C is $\alpha_i^\vee \in H_2(X, \mathbb{Z})$. Here, $H_2(X, \mathbb{Z})$ is identified with the lattice generated by the coroots corresponding to elements of I. Therefore $\coprod_{\alpha_i\in I} \mathbf{R}_{\alpha_i^\vee}(X)$ parametrizes $\mathcal{L}_I$-lines on X.

Theorem 2.2.7 ([23, Theorem 4.3 and Theorem 4.8])**.** *Let $X = G/P_I$ be a rational homogeneous space, G being a simply connected simple Lie group and nonempty $I \subset S$. Assume that $\alpha_i \in I$ is long (as a root), and let $N(\alpha_i) := \{\alpha_j \in S : \langle \alpha_i, \alpha_j \rangle < 0\}$. Then the following hold.*

1. *G acts transitively on $\mathbf{R}_{\alpha_i^\vee}(X)$, and $\mathbf{R}_{\alpha_i^\vee}(X) \simeq G/P_{(I\setminus\{\alpha_i\})\cup N(\alpha_i)}$.*

2. *For the base point $o := e\cdot P_I \in X$, P_I acts transitively on*

$$\mathcal{C}_o^{\alpha_i} := \{[T_oC] \in \mathbb{P}(T_oX) : [C] \in \mathbf{R}_{\alpha_i^\vee}(X) \text{ such that } o \in C\}.$$

Note that $\mathcal{C}_o^{\alpha_i}$ is projective, hence for the standard Levi subgroup L_I of P_I, $\mathcal{C}_o^{\alpha_i} \simeq L_I/(L_I \cap P_I)$.

As a corollary, let us describe the space of lines on the adjoint variety $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ for each simple Lie algebra $\mathfrak{g}$ not of type C. (In fact, if $\mathfrak{g}$ is of type C, then since $\rho = 2\omega_1$, $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ is the second Veronese embedding of the projective space, hence $Z_{\mathfrak{g}}$ does not contain any line on $\mathbb{P}(\mathfrak{g})$.) If $\mathfrak{g}$ is not of type C, we have $\rho = \sum_{\alpha_i\in N(\rho)} \omega_i$ (Table 2.1), which means that $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}} \simeq \mathcal{L}_{N(\rho)}$. That is, lines on $Z_{\mathfrak{g}}$ with respect to the embedding into $\mathbb{P}(\mathfrak{g})$ are exactly $\mathcal{L}_{N(\rho)}$-lines.

- If $\mathfrak{g}$ is not of type A or C, then $N(\rho) = \{\alpha_{j_0}\}$ for some $\alpha_{j_0} \in S$, hence the space of lines on $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ is isomorphic to $G/P_{N(\alpha_{j_0})}$. On the other hand, by [23, Proposition 2.4], $D_o \subset T_oZ_{\mathfrak{g}}$ is an irreducible P-module. By Theorem 2.2.7, $\mathcal{C}_o = \mathcal{C}_o^{\alpha_{j_0}}$ is the highest weight orbit in $\mathbb{P}(D_o)$. (This description can be also found in [15, Proposition 5].)

- If $\mathfrak{g} = A_r$ $(r \geq 2)$, then the space of lines has two connected components $\mathbf{R}_{\alpha_1^\vee}(X) \sqcup \mathbf{R}_{\alpha_r^\vee}(X)$ where

$$\mathbf{R}_{\alpha_1^\vee}(X) \simeq G/P_{\alpha_2,\alpha_r}, \quad \text{and} \quad \mathbf{R}_{\alpha_r^\vee}(X) \simeq G/P_{\alpha_1,\alpha_{r-1}}.$$

  Moreover, we have $D_o = D_o^{\alpha_1} \oplus D_o^{\alpha_r}$ as P-modules where under the identification $D_o \simeq \mathfrak{g}^{-1}/\mathfrak{p}$,

$$D_o^{\alpha_1} := \left(\bigoplus_{1\leq i<r} \mathfrak{g}_{-(\alpha_1+\cdots+\alpha_i)} \oplus \mathfrak{p}\right)/\mathfrak{p}, \quad \text{and} \quad D_o^{\alpha_r} := \left(\bigoplus_{1<i\leq r} \mathfrak{g}_{-(\alpha_i+\cdots+\alpha_r)} \oplus \mathfrak{p}\right)/\mathfrak{p}.$$

  By [23, §2.3], each of $D_o^{\alpha_1}$ and $D_o^{\alpha_r}$ is an irreducible P-module. By Theorem 2.2.7, for each $i = 1, r$, $\mathcal{C}_o^{\alpha_i}$ is the highest weight orbit in $\mathbb{P}(D_o^{\alpha_{i'}})$ for some $i' = 1, r$. In fact, for the projection $Z_{\mathfrak{g}} = G/P_{\alpha_1,\alpha_r} \to G/P_{\alpha_1}$, $D_o^{\alpha_r}$ is the tangent space of a fiber, hence the lines tangent to $D_o^{\alpha_r}$ are contained in the fiber. Therefore $\mathcal{C}_o^{\alpha_i}$ is the highest weight orbit in $\mathbb{P}(D_o^{\alpha_i})$ for $i = 1, r$.

The space of lines passing through o on $Z_{\mathfrak{g}}$ is described in Table 2.2. Here, $\nu_k(\mathbb{P}^1)$ means the kth Veronese embedding of $\mathbb{P}^1$. Observe that the dimension of the space of lines through o is always equal to $n-1$.

2.2.3 Spaces of Complete Conics

In contrast to the case of lines, the spaces of smooth rational curves of higher degree are not projective. In this subsection, we consider smooth conics, i.e. smooth rational curves of degree 2, and introduce the notion of the space of complete conics, which is a compactification of the space of smooth conics.

First, we recall the construction of the space of complete conics on $\mathbb{P}^2$. Let V be a vector space of dimension 3. Then $\mathbb{P}(\mathrm{Sym}^2V^*)$, isomorphic to $\mathbb{P}^5$, parametrizes hyperquadrics on $\mathbb{P}(V)$. Indeed, with respect to the natural $PGL(V)$-action, $\mathbb{P}(\mathrm{Sym}^2V^*)$ consists of three orbits:

- The locus of smooth conics. This is an open $PGL(V)$-orbit, isomorphic to $PGL(V)/PO(V)$. Here, $PO(V) := O(V)/\pm id$ denotes the image of the orthogonal group $O(V)$ in $PGL(V)$.

- The locus of reducible conics, i.e. unions of two lines. This orbit is of codimension 1, but it is not a closed $PGL(V)$-orbit.

$\mathfrak{g}$	Description of $\bigsqcup_{\alpha_i \in N(\rho)} \mathcal{C}_o^{\alpha_i} \subset \mathbb{P}(D_o)$
A_r $(r \geq 2)$	$\mathbb{P}^{r-2} \sqcup \mathbb{P}^{r-2}$ (disjoint linear subspaces)
B_r $(r \geq 4)$	$\mathbb{P}^1 \times \mathbb{Q}^{2r-5}$ (Segre)
D_r $(r \geq 5)$	$\mathbb{P}^1 \times \mathbb{Q}^{2r-6}$ (Segre)
B_3	$\mathbb{P}^1 \times \nu_2(\mathbb{P}^1)$ (Segre)
D_4	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Segre)
E_6	Gr(3, 6) (Plücker)
E_7	OG(6, 12) (Plücker)
E_8	E_7/P_1 (minimal embedding)
F_4	LG(3, 6) (minimal embedding)
G_2	$\nu_3(\mathbb{P}^1)$

Table 2.2: Space of lines passing through $o \in Z_{\mathfrak{g}}$ and its embedding into the contact hyperplane.

- The locus of double lines, i.e. non-reduced hyperquadrics. This is a unique closed $PGL(V)$-orbit, isomorphic to $\mathrm{Gr}(2, V)$ (of codimension 3).

Then the *space of complete conics* on $\mathbb{P}(V)$ is defined to be the blow-up of $\mathbb{P}(\mathrm{Sym}^2 V^*)$ along the orbit of double lines. We denote it by $\mathbf{CoC}(\mathbb{P}(V))$.

For arbitrary projective subvariety $X \subset \mathbb{P}^N$, $N \geq 2$, we recall the following well-known fact:

Proposition 2.2.8 ([30, Remark 4.4.(i)])**.** *Let $C \subset \mathbb{P}^N$ be a closed subscheme with Hilbert polynomial $2m+1$. Then there exists a unique plane in $\mathbb{P}(\mathfrak{g})$ which contains C as a closed subscheme. In particular, the scheme C is isomorphic to a hyperquadric on a plane.*

Definition 2.2.9. Let $X \subset \mathbb{P}^N$ be a projective subvariety.

1. A closed subscheme $C \subset X$ is called a *conic* if its Hilbert polynomial in $\mathbb{P}^N$ is equal to $2m+1$.
2. A conic $C \subset X$ is called a *reducible conic* (a *double line*, respectively) if C is the union of two distinct lines (C is non-reduced, respectively).

Let us write $\mathbb{P}^N = \mathbb{P}(W)$ for a vector space W of dimension $N+1$. Choose a 3-dimensional subspace $V \subset W$ so that $\mathrm{Gr}(3, W) \simeq PGL(W)/\mathrm{Stab}_{PGL(W)}(V)$. By Proposition 2.2.8, the Hilbert scheme of conics on $\mathbb{P}(W)$ is given by

$$\mathrm{Hilb}_{2m+1}(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \simeq PGL(W) \times_{\mathrm{Stab}_{PGL(W)}(V)} \mathbb{P}(\mathrm{Sym}^2 V^*),$$

a homogeneous fiber bundle over $\mathrm{Gr}(3, W)$ with fiber $\simeq \mathbb{P}(\mathrm{Sym}^2 V^*)$. Then we define the *space of complete conics* on $\mathbb{P}(W)$ as

$$\mathbf{CoC}(\mathbb{P}(W)) := PGL(W) \times_{\mathrm{Stab}_{PGL(W)}(V)} \mathbf{CoC}(\mathbb{P}(V)).$$

By its definition, $\mathbf{CoC}(\mathbb{P}(W))$ contains the space of smooth conics (and reducible conics) on $\mathbb{P}(W)$. Thus we may define the *space of complete conics* on $X (\subset \mathbb{P}(W))$ as

$$\mathbf{CoC}(X, \mathcal{O}_{\mathbb{P}(W)}(1)|_X) := \overline{\{\text{smooth conics on } X \subset \mathbb{P}(W)\}} \subset \mathbf{CoC}(\mathbb{P}(W)).$$

Observe that by the definitions, we have a $\mathrm{Stab}_{PGL(W)}(V)$-equivariant morphism (the blowing-up)

$$\mathbf{CoC}(\mathbb{P}(V)) \to \mathbb{P}(\mathrm{Sym}^2 V^*),$$

which induces a $PGL(W)$-equivariant birational morphism

$$\mathbf{CoC}(\mathbb{P}(W)) \to \mathrm{Hilb}_{2m+1}(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)).$$

This is an isomorphism over $\mathrm{Hilb}_{2m+1}(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \setminus \{\text{double lines}\}$. Thus it induces a morphism

$$CH : \mathbf{CoC}(X, \mathcal{O}_{\mathbb{P}(W)}(1)|_X) \to \overline{\{\text{smooth conics}\}} \subset (\mathrm{Hilb}_{2m+1}(X, \mathcal{O}_{\mathbb{P}(W)}(1)|_X))^{red}$$

which is an isomorphism off the locus of double lines.

Remark 2.2.10. 1. Alternatively, the space $\mathbf{CoC}(\mathbb{P}(V))$ of complete conics on the plane $\mathbb{P}(V)$ ($\dim V = 3$) can be constructed as the closure of

$$\{([C], [C^\vee]) : C \subset \mathbb{P}(V) \text{ smooth conics}\} \subset \mathbb{P}(\mathrm{Sym}^2 V^*) \times \mathbb{P}(\mathrm{Sym}^2 V).$$

Here, for a smooth conic $C \subset \mathbb{P}(V)$, $C^\vee \subset \mathbb{P}(V^*)$ is the dual hypersurface, i.e. the set of points corresponding to lines tangent to C (and it is known that $C^\vee$ is again a smooth conic). In this construction, the defining equation of $\mathbf{CoC}(\mathbb{P}(V))$ can be described as follows. First, choose a basis of V so that $\mathbb{P}(\mathrm{Sym}^2 V^*)$ is identified with the projectivization of the set $\mathbb{M}_3$ of 3×3 symmetric matrices. Similarly, its dual basis induces the identification between $\mathbb{P}(\mathrm{Sym}^2 V)$ and $\mathbb{P}\mathbb{M}_3$. Then $\mathbf{CoC}(\mathbb{P}(V))$ can be identified with the set of $([M_1], [M_2]) \in \mathbb{P}\mathbb{M}_3 \times \mathbb{P}\mathbb{M}_3$ such that $M_1 \cdot M_2$ is a scalar matrix (possibly zero). See [42, §5] and [41, Example 17.12].

2. The space $\mathbf{CoC}(\mathbb{P}^N)$ of complete conics on $\mathbb{P}^N$ ($N \geq 2$) is a *wonderful variety*. Here, for a connected reductive group G', a smooth projective G'-variety $\mathbf{W}$ is called *wonderful* if

   (a) $\mathbf{W}$ contains an open G'-orbit, and its complement is the union of prime divisors $\mathbf{W}_i$, $1 \leq i \leq r$ such that each $\mathbf{W}_i$ is smooth and $\mathbf{W}_i$ and $\mathbf{W}_j$ intersect transversally for all $i \neq j$; and

   (b) For arbitrary $\varnothing \neq I \subset \{1, \ldots, r\}$,

   $$\bigcap_{i \in I} \mathbf{W}_i \setminus \bigcup_{j \notin I} \mathbf{W}_j$$

   is a single G'-orbit.

   Then $\mathbf{CoC}(\mathbb{P}^N)$ is a wonderful PGL_{N+1}-variety with $r = 2$. See [7, Exemple 2.7.(b)].

2.3 Luna-Vust Theory for Symmetric Varieties

2.3.1 Spherical Varieties

Let us review the embedding theory of spherical varieties. Our main reference is [41] and [19]. Let G' be a connected reductive group. A normal G'-variety X is called *(G'-)spherical* if a Borel subgroup has an open orbit in X, or equivalently if a Borel subgroup has only finitely many orbits. If O is the open G'-orbit in the spherical variety X, then O is a homogeneous spherical variety, and we say that X *is an O-embedding.*

Remark 2.3.1. Here are several examples of spherical varieties.

1. If G' is a torus, then the notion of G'-spherical varieties coincides with that of toric varieties, since a torus is a Borel subgroup of itself.

2. Rational homogeneous spaces G'/P' are G'-spherical. This is a consequence of the Bruhat decomposition.

3. Wonderful G'-varieties (Remark 2.2.10) are G'-spherical varieties. This is a result of Luna ([41, Theorem 30.15]).

Luna-Vust theory ([27]) says that given a homogeneous spherical variety O, O-embeddings can be classified in terms of combinatorial data, called *colored fans*, which can be constructed as follows. Let O be a homogeneous G'-spherical variety, $T' \subset G'$ a maximal torus, and $B' \subset G'$ a Borel subgroup containing T'. For the set $\mathbb{C}(O)^{(B')} \subset \mathbb{C}(O)^{\times}$ of rational B'-eigenfunctions, define

$$\Lambda_O := \{\chi \in \chi(B') : b.f = \chi(b) \cdot f \quad \forall b \in B' \text{ for some } f \in \mathbb{C}(O)^{(B')}\},$$

which is a sublattice of the character group $\chi(B') \simeq \chi(T')$. If $\mathbb{C}(O)^{B'} \subset \mathbb{C}(O)^{\times}$ denotes the set of B'-invariant rational functions on O, then since we have a short exact sequence

$$0 \to \mathbb{C}^{\times} = \mathbb{C}(O)^{B'} \to \mathbb{C}(O)^{(B')} \to \Lambda_O \to 0,$$

a valuation $v : \mathbb{C}(O)^{\times} \to \mathbb{Q}$ induces a group homomorphism $\Lambda_O \to \mathbb{Q}$. In other words, a valuation corresponds to an element in the $\mathbb{Q}$-vector space $\mathcal{E} := \mathrm{Hom}_{\mathbb{Z}}(\Lambda_O, \mathbb{Q})$. Moreover, this correspondence is injective for G'-invariant valuations ([19, Corollary 2.8]), hence we may identify the set of G-invariant valuations on O with its image in $\mathcal{E}$, denoted by $\mathcal{V}$ and called the *valuation cone*. If we define

$$\mathcal{D}(O) := \{B'\text{-stable prime Weil divisors of } O\}$$

and consider the valuation induced by each element of $\mathcal{D}(O)$, then a similar process yields a function $\epsilon : \mathcal{D}(O) \to \mathcal{E}$, which is not injective in general. The elements of $\mathcal{D}(O)$ are called *colors*.

Next, consider a *simple* O-embedding X, meaning that X is an O-embedding which contains exactly one closed G'-orbit. If $Y \subset X$ is the unique closed orbit in X, define

$$\mathcal{F}(X) := \{D \in \mathcal{D}(O) : Y \subset \overline{D} \text{ in } X\}.$$

We call each element of $\mathcal{F}(X)$ a *color* of X. Since G'-stable prime divisors of X can be considered as elements of $\mathcal{V}$, it is possible to define a convex cone in $\mathcal{E}$ by

$$\mathcal{C}(X) := \mathbb{Q}_{\geq 0}\langle \epsilon(\mathcal{F}(X)), G'\text{-stable prime divisors of } X\rangle.$$

Now let X be an arbitrary O-embedding. For every G'-orbit $Y \subset X$, Y has a G'-stable open neighborhood

$$X_Y := \{x \in X : Y \subset \overline{G' \cdot x}\}$$

which is a simple O-embedding such that Y is its unique closed orbit. Thus to X, we can associate a collection of pairs

$$\mathfrak{F}(X) := \{(\mathcal{C}(X_Y), \mathcal{F}(X_Y)) : Y \text{ is a } G'\text{-orbit in } X\}.$$

Then each $(\mathcal{C}(X_Y), \mathcal{F}(X_Y))$ is a *colored cone*, and $\mathfrak{F}(X)$ is a *colored fan*, in the following sense:

Definition 2.3.2. Let $\mathcal{E}_0$ be a finite dimensional $\mathbb{Q}$-vector space, $\mathcal{D}_0$ a finite set, $\mathcal{V}_0 \subset \mathcal{E}_0$ a convex cone, and $\epsilon_0 : \mathcal{D}_0 \to \mathcal{E}_0$ a function.

1. A *colored cone* for $(\mathcal{E}_0, \mathcal{D}_0, \mathcal{V}_0, \epsilon_0)$ is a pair $(\mathcal{C}, \mathcal{F})$ of subsets $\mathcal{C} \subset \mathcal{E}_0$ and $\mathcal{F} \subset \mathcal{D}_0$ such that

(a) $\mathcal{C}$ is a convex cone generated by $\epsilon_0(\mathcal{F})$ and finitely many elements in $\mathcal{V}_0$; and

(b) the relative interior of $\mathcal{C}$ intersects with $\mathcal{V}_0$.

2. A colored cone $(\mathcal{C}, \mathcal{F})$ is called *strictly convex* if $\mathcal{C}$ is strictly convex and $0 \notin \epsilon_0(\mathcal{F})$.

3. For colored cones $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{F}')$, $(\mathcal{C}', \mathcal{F}')$ is called a *colored face* of $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C}'$ is a face of the cone $\mathcal{C}$ and $\mathcal{F}' = \mathcal{F} \cap \epsilon_0^{-1}(\mathcal{C}')$.

4. A nonempty finite set $\mathfrak{F}$ of colored cones for $(\mathcal{E}_0, \mathcal{D}_0, \mathcal{V}_0, \epsilon_0)$ is called a *colored fan* if

(a) For every element of $\mathfrak{F}$, its colored faces are contained in $\mathfrak{F}$; and

(b) For every $v \in \mathcal{V}_0$, there is at most one element of $\mathfrak{F}$ of which relative interior contains v.

5. A colored fan is called *strictly convex* if it consists of strictly convex colored cones.

Keeping the previous notation, for a homogeneous spherical variety O, a colored cone/fan for $(\mathcal{E}, \mathcal{D}(O), \mathcal{V}, \epsilon)$ is called a *colored cone/fan* for O.

Theorem 2.3.3 ([19, Theorem 4.3], [41, Section 15])**.** *For a homogeneous spherical variety O, the map $X \mapsto \mathfrak{F}(X)$ is a bijection between isomorphism classes of O-embeddings, and strictly convex colored fans for O.*

Under this correspondence, a simple O-embedding X is corresponding to a colored fan consisting of $(\mathcal{C}(X), \mathcal{F}(X))$ and its colored faces. Conversely, every strictly convex colored cone is induced from a simple O-embedding.

A lot of geometric properties of spherical varieties can be expressed in terms of colored data. For example, we have the following lemmas.

Lemma 2.3.4 ([19, Lemma 4.2])**.** *For a G'-spherical variety X, the assignment $Y \mapsto (\mathcal{C}(X_Y), \mathcal{F}(X_Y))$ between G'-orbits in X and elements of $\mathfrak{F}(X)$ is bijective and order-reversing. Here, the set of orbits is (partially) ordered by inclusion of closures.*

Lemma 2.3.5 ([19, Theorem 5.2])**.** *A spherical variety X is complete if and only if the valuation cone $\mathcal{V}$ is contained in the union of colored cones in the colored fan of X. In particular, if X is simple, then X is a complete variety if and only if $\mathcal{C}(X)$ is generated by $\epsilon(\mathcal{F}(X))$ and $\mathcal{V}$.*

Lemma 2.3.6 ([19, Lemma 7.5])**.** *Let O be a homogeneous G'-spherical variety. Suppose that X is a simple O-embedding and its unique closed orbit Y is projective. Let B' be a Borel subgroup containing a maximal torus T', $T'' := g^{-1} \cdot T' \cdot g$ and $B'' := g^{-1} \cdot B' \cdot g$ for some $g \in G'$, and w_0 a representative of the longest element in the Weyl group of (G', T'') with respect to B''. Then the stabilizer of the unique B''-fixed point in Y is the opposite parabolic subgroup (containing B'') of*

$$\bigcap_{\mathcal{D} \in \mathcal{D}(O) \setminus \mathcal{F}(X)} w_0 \cdot g^{-1} \cdot Stab_{G'}(\mathcal{D}) \cdot g \cdot w_0^{-1}$$

where the colored data $\mathcal{D}(O)$ and $\mathcal{F}(X)$ are defined with respect to B'.

Non-normal embeddings of a homogeneous spherical variety can be studied by the following proposition:

Proposition 2.3.7 ([41, Proposition 15.15])**.** *For a G-variety X admitting a G-linearized ample line bundle, if X contains an open G-orbit which is spherical, then its normalization map $\pi : X^{nor} \to X$ is bijective on the sets of G-orbits. That is, for a G-orbit $\mathcal{O}$ of X, $\pi^{-1}(\mathcal{O})$ is also a single G-orbit.*

Remark 2.3.8. Proposition 2.3.7 does not mean that the orbits are isomorphic as varieties. However, since π is a finite birational morphism, $\mathcal{O}$ and $\pi^{-1}(\mathcal{O})$ are isomorphic if $\mathcal{O}$ is either open or projective.

2.3.2 Symmetric Varieties

From now on, we focus on symmetric varieties.

Definition 2.3.9. For a connected reductive group G' and its closed subgroup K', a homogeneous variety G'/K' is called *(G'-)symmetric* if there is a nontrivial involution $\sigma : G' \to G'$ such that $(G')^\sigma \subset K' \subset N_{G'}((G')^\sigma)$ where $(G')^\sigma$ is the fixed point subgroup. A normal G'-variety which contains an open G'-orbit isomorphic to a homogeneous symmetric variety is also called *(G'-)symmetric*.

A homogeneous G'-symmetric variety is G'-spherical ([41, Theorem 26.14], [11, Section 1.3]), and so Theorem 2.3.3 can be applied to symmetric varieties. Indeed, Vust ([43]) obtains a practical description of colored data for symmetric varieties, which is explained in this subsection, following [41, Section 26].

For simplicity, from now on, we assume that G' is a simply connected semi-simple Lie group and $K' := (G')^\sigma$ for an involution $\sigma(\neq id)$. This assumption implies that $K' = (G')^\sigma$ is a connected reductive subgroup ([39, Section 8]). Put $O := G'/K'$, and let T' be a maximally σ-split torus in G'. That is, T' is a maximal torus such that T' is σ-stable and $\dim\{t \in T' : \sigma(t) = t^{-1}\}$ is maximal among all maximal tori. As before, B' is a Borel subgroup containing T'. Then define R' and S' as the root system and the set of simple roots defined by (G', T', B'), respectively. Let T'_1 be the identity component of $\{t \in T' : \sigma(t) = t^{-1}\}$ so that T'_1 is a subtorus of T'. Consider subsets

$$R'_O := \{\overline{\alpha'} \in \chi(T'_1) : \alpha' \in R'\} \setminus \{0\},$$
$$S'_O := \{\overline{\alpha'_i} \in \chi(T'_1) : \alpha'_i \in S'\} \setminus \{0\}$$

where $\overline{\alpha'} := \alpha'|_{T'_1}$. It is well-known that one can choose B' so that for every positive root $\alpha' \in R'$ such that $\overline{\alpha'} \neq 0$, we have $\sigma(\alpha') < 0$, under the natural action of σ on R' ([11, Lemma 1.2]; this condition ensures that $B' \cdot K'/K'$ is open in O). Then R'_O becomes a root system of $\chi(T'_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ with simple roots in S'_O ([41, Lemma 26.16]), called the *restricted root system*. Moreover, the lattice Λ_O is isomorphic to the character group $\chi(T'/T' \cap K') = \chi(T'_1/T'_1 \cap K')$. This lattice is a sublattice of $\chi(T'_1)$ with finite index, thus the vector space $\mathcal{E}$ is identified with $\chi_*(T'_1) \otimes \mathbb{Q}$.

Observe that the restriction $\chi(T') \otimes_{\mathbb{Z}} \mathbb{Q} \to \chi(T'_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with the orthogonal projection

$$\chi(T') \otimes_{\mathbb{Z}} \mathbb{Q} \to \chi(T') \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \chi \mapsto \frac{\chi - \sigma(\chi)}{2}.$$

By identifying its image with $\chi(T'_1) \otimes_{\mathbb{Z}} \mathbb{Q}$, the dual root system $(R'_O)^\vee$ can be realized as follows. For any $\alpha' \in R'$ satisfying $\overline{\alpha'} \neq 0$, Vust [43, Lemme 2.3] shows that one of the following holds:

- $\sigma(\alpha') = -\alpha'$. In this case, put $\overline{\alpha'}^V := (\alpha')^\vee$. Then for all $\chi \in \chi(T)$, we have
$$\langle \overline{\alpha'}^V, \overline{\chi} \rangle = \langle (\alpha')^\vee, \frac{\chi - \sigma(\chi)}{2} \rangle = \langle \chi \,|\, \alpha' \rangle.$$
- $\langle (\alpha')^\vee, \sigma(\alpha') \rangle = 0$. In this case, put $\overline{\alpha'}^V := (\alpha')^\vee - \sigma(\alpha')^\vee$. Then for all $\chi \in \chi(T)$, we have
$$\langle \overline{\alpha'}^V, \overline{\chi} \rangle = \langle (\alpha')^\vee - \sigma(\alpha')^\vee, \frac{\chi - \sigma(\chi)}{2} \rangle = \langle \chi \,|\, \alpha' \rangle - \langle \chi \,|\, \sigma(\alpha') \rangle.$$
- $\langle (\alpha')^\vee, \sigma(\alpha') \rangle = 1$. In this case, put $\overline{\alpha'}^V := 2((\alpha')^\vee - \sigma(\alpha')^\vee)$. Then for all $\chi \in \chi(T)$, we have
$$\langle \overline{\alpha'}^V, \overline{\chi} \rangle = 2\langle (\alpha')^\vee - \sigma(\alpha')^\vee, \frac{\chi - \sigma(\chi)}{2} \rangle = 2(\langle \chi \,|\, \alpha' \rangle - \langle \chi \,|\, \sigma(\alpha') \rangle).$$

Then $\langle \overline{\alpha'}^V, \overline{\alpha'} \rangle = 2$ for all $\overline{\alpha'} \in R'_O$, and $(R'_O)^\vee = \{\overline{\alpha'}^V : \overline{\alpha'} \in R'_O\}$.

Its base $(S'_O)^\vee$, called the set of *restricted simple coroots*, can be obtained from the *Satake diagram*, which encodes information on the action of σ on R'. Satake diagrams play an important role in the classification of homogeneous symmetric varieties. For details and the list of all possible Satake diagrams arising from simple Lie groups, we refer to [41, §26.5] and [36, Table 1]. The Satake diagram of σ can be constructed as follows.

1. Start with the Dynkin diagram of G'.
2. For every simple root (with respect to B' and T' as before) which is σ-stable, mark the corresponding node by black.
3. Mark the nodes corresponding to σ-unstable simple roots by white.
4. If two σ-unstable simple roots $\alpha'_i \neq \alpha'_j$ satisfy $\overline{\alpha'_i} = \overline{\alpha'_j}$, then join the corresponding (white) nodes by a two-headed arrow.

Now put $(S'_O)^\vee := \{\lambda^\vee : \lambda \in S'_O\}$ where for each $\lambda = \overline{\alpha'} \in S'_O$, with a slight abuse of notation, $\lambda^\vee$ is defined as follows:

1. In the Satake diagram, if α' represents a white node which is not joined by an arrow and not adjacent to a black node, put $\lambda^\vee := (\alpha')^\vee$. (This is the case exactly when $\sigma(\alpha') = -\alpha'$.)
2. Otherwise, put $\lambda^\vee := (\alpha')^\vee - \sigma(\alpha')^\vee$.

See [41, Remark 26.23].

Remark 2.3.10. If R'_O is reduced, then there is no $\alpha' \in R'$ with $\langle (\alpha')^\vee, \sigma(\alpha') \rangle = 1$, hence $\lambda^V = \lambda^\vee$ for all $\lambda \in S'_O$.

Theorem 2.3.11 ([41, Section 26], [43, Section 2.4], [37, Section 2])**.** *In the previous notation, via the isomorphism $\mathcal{E} \simeq \chi_*(T'_1) \otimes \mathbb{Q}$, we have the following identifications:*

1. *The lattice Λ_O is identified with the doubled weight lattice $2 \cdot (\mathbb{Z}\langle (R'_O)^\vee \rangle)^* \subset \chi(T'_1) \otimes \mathbb{Q}$.*
2. *The image $\epsilon(\mathcal{D}(O))$ in $\mathcal{E}$ is exactly $\frac{1}{2}(S'_O)^\vee$.*
3. *$\mathcal{V}$ is identified with the negative Weyl chamber of $(R'_O)^\vee$ in $\mathcal{E}$.*

Moreover, if K' is semi-simple, then the map $\epsilon : \mathcal{D}(O) \to \mathcal{E}$ is injective. In this case, if $\mathcal{D} \in \mathcal{D}(O)$ is sent to $\frac{1}{2}\lambda^\vee$ for some $\lambda \in S'_O$, then the stabilizer $Stab_{G'}(\mathcal{D})$ of the divisor $\mathcal{D} \subset O$ is

$$Stab_{G'}(\mathcal{D}) = P'_{\{\alpha'_j \in S' : \overline{\alpha'_j} = \lambda\}},$$

i.e. the parabolic subgroup containing B' and generated by simple roots $\alpha'_k \in S'$ such that either $\overline{\alpha'_k} = 0$ or $\overline{\alpha'_k} \in S'_O \setminus \{\lambda\}$.

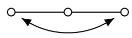
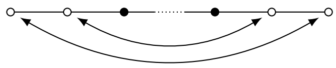
Remark 2.3.12. In general, K' is not necessarily semi-simple, and it is possible that the color map ϵ is not injective. Namely, if G' is simple, K' is not semi-simple and $\lambda^\vee \in (S'_O)^\vee$ is short (i.e. its length is the minimum among $(R'_O)^\vee$), then $\epsilon^{-1}(\lambda^\vee/2)$ consists of two colors by [41, p. 157]. For example, consider $G' = SL_{r+1}$ ($r \geq 2$) and an involution $\sigma : G' \to G'$ defined as

$$\sigma(g) := I_2 \cdot g \cdot I_2, \quad I_2 := \begin{pmatrix} -id_{2\times 2} & 0 \\ 0 & id_{(r-1)\times(r-1)} \end{pmatrix}.$$

Then

$$K' = (G')^\sigma = \left\{ \begin{pmatrix} M_2 & 0 \\ 0 & M_{r-1} \end{pmatrix} \in SL_{r+1} : M_i \in GL_i,\ i = 2,\ r-1 \right\} = S(GL_2 \times GL_{r-1}).$$

Since K' contains a maximal torus of G' while its semi-simple part ($= SL_2 \times SL_{r-1}$) is of rank $r-1$, K' is not semi-simple. The Satake diagram of O is

 if $r = 2$,  if $r = 3$, and  if $r \geq 4$,

and the restricted root system R'_O is BC_1 if $r = 2$, C_2 if $r = 3$, and BC_2 if $r \geq 4$. See [36, Table 1]. In the following, we describe ϵ and the stabilizer of each color:

- If $r = 2$, then since $R'_O = BC_1$, $(S'_O)^\vee = \{\lambda^\vee\}$ consists of a short coroot. Thus by [41, p. 157], $\mathcal{D}(O) = \{\mathcal{D}_1, \mathcal{D}_2\}$ such that $\epsilon(\mathcal{D}_1) = \epsilon(\mathcal{D}_2) = \lambda^\vee/2$. Moreover, by [37, p. 151–152], up to rearrangement, we have
$$\mathrm{Stab}_{G'}(\mathcal{D}_j) = P'_{\alpha'_j}, \quad j = 1, 2.$$

- If $r = 3$, then $R'_O = C_2$. Put $\lambda_1 := \overline{\alpha'_1} = \overline{\alpha'_3}$ and $\lambda_2 := \overline{\alpha'_2}$, the elements of S'_O. In this case, we have $\sigma(\alpha'_1) = -\alpha'_3$ and $\sigma(\alpha'_2) = -\alpha'_2$. Thus $\lambda_1^\vee = (\alpha'_1)^\vee + (\alpha'_3)^\vee$ and $\lambda_2^\vee = (\alpha'_2)^\vee$, hence $\langle \lambda_1^\vee \,|\, \lambda_2^\vee \rangle = -2$. It means that $(S'_O)^\vee$ consists of $\lambda_1^\vee$ (long) and $\lambda_2^\vee$ (short). By [41, p. 157], we have $\mathcal{D}(O) = \{\mathcal{D}_1, \mathcal{D}_2^+, \mathcal{D}_2^-\}$ such that
$$\epsilon(\mathcal{D}_1) = \frac{1}{2}\lambda_1^\vee, \quad \epsilon(\mathcal{D}_2^\pm) = \frac{1}{2}\lambda_2^\vee,$$
and by [37, p. 151–152],
$$\mathrm{Stab}_{G'}(\mathcal{D}_1) = P'_{\{\alpha'_j \in S' : \overline{\alpha'_j}^\vee = \lambda_1^\vee\}} (= P'_{\alpha'_1, \alpha'_3}), \quad \mathrm{Stab}_{G'}(\mathcal{D}_2^\pm) = P'_{\alpha'_2}.$$

- If $r \geq 4$, then $R'_O = BC_2$. Put $\lambda_1 := \overline{\alpha'_1} = \overline{\alpha'_r}$ and $\lambda_2 := \overline{\alpha'_2} = \overline{\alpha'_{r-1}}$. Then using [11, Lemma 1.4], one can check that $\sigma(\alpha'_1) = -\alpha'_r$, while $\langle (\alpha'_2)^\vee, \sigma(\alpha'_2) \rangle = 1$. Thus we have
$$\langle \lambda_1^\vee \,|\, \lambda_2^\vee \rangle = \langle \lambda_1^V \,|\, \lambda_2^V/2 \rangle = 2 \cdot \langle \lambda_2, \lambda_1^V \rangle = -2.$$
That is,
$$(S'_O)^\vee = \{\lambda_1^\vee \text{ (not short)}, \lambda_2^\vee \text{ (short)}\}.$$
By [41, p. 157], we have $\mathcal{D}(O) = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_{r-1}\}$ such that
$$\epsilon(\mathcal{D}_1) = \frac{1}{2}\lambda_1^\vee, \quad \epsilon(\mathcal{D}_2) = \epsilon(\mathcal{D}_{r-1}) = \frac{1}{2}\lambda_2^\vee,$$
and by [37, p. 151–152],
$$\mathrm{Stab}_{G'}(\mathcal{D}_1) = P'_{\{\alpha'_j \in S' : \overline{\alpha'_j} = \lambda_1^\vee\}} (= P'_{\alpha'_1, \alpha'_r}), \quad \mathrm{Stab}_{G'}(\mathcal{D}_2) = P'_{\alpha'_2}, \quad \mathrm{Stab}_{G'}(\mathcal{D}_{r-1}) = P'_{\alpha'_{r-1}}.$$

Chapter 3. Geometry of Conics on Adjoint Varieties

In this chapter, we study geometry of conics on $Z_{\mathfrak{g}}$. Namely, we show that there is an open G-orbit in the space of smooth conics on $Z_{\mathfrak{g}}$, which can be described in terms of the contact distribution. Then we prove that it is a G-symmetric variety, and then study B-fixed points in its boundary. From now on, we assume that $\dim Z_{\mathfrak{g}} > 1$ (i.e. $\mathfrak{g} \neq A_1$, or equivalently $Z_{\mathfrak{g}} \not\cong \mathbb{P}^1$).

3.1 Conics on Adjoint Varieties

Our goal is to study deformation of smooth conics on the adjoint variety $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$. Note that the family of smooth conics on $Z_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ is parametrized by

$$\mathbf{R}_2(Z_{\mathfrak{g}}) := \coprod_{\check{\alpha}} \mathbf{R}_{\check{\alpha}}(Z_{\mathfrak{g}})$$

where $\check{\alpha}$ runs over elements of form $\sum_{\alpha_i \in N(\rho)} m_i \cdot \alpha_i^{\vee}$ such that $m_i \in \mathbb{Z}_{\geq 0}$ and $\langle \rho, \check{\alpha} \rangle = 2$. Namely, if $\mathfrak{g}$ is not of type A, then since $N(\rho) = \{\alpha_{j_0}\}$,

$$\mathbf{R}_2(Z_{\mathfrak{g}}) = \mathbf{R}_{(2/\langle \rho | \alpha_{j_0} \rangle) \cdot \alpha_{j_0}^{\vee}}(Z_{\mathfrak{g}}) = \begin{cases} \mathbf{R}_{2 \cdot \alpha_{j_0}^{\vee}}(Z_{\mathfrak{g}}) & \text{if } \mathfrak{g} \text{ is not of type } C, \\ \mathbf{R}_{\alpha_1^{\vee}}(Z_{\mathfrak{g}}) & \text{if } \mathfrak{g} = C_r,\ r \geq 2. \end{cases}$$

On the other hand, if $\mathfrak{g} = A_r$, $r \geq 2$, then since $N(\rho) = \{\alpha_1, \alpha_r\}$,

$$\mathbf{R}_2(Z_{\mathfrak{g}}) = \mathbf{R}_{2\alpha_1^{\vee}}(Z_{\mathfrak{g}}) \sqcup \mathbf{R}_{\alpha_1^{\vee} + \alpha_r^{\vee}}(Z_{\mathfrak{g}}) \sqcup \mathbf{R}_{2\alpha_r^{\vee}}(Z_{\mathfrak{g}}).$$

By taking closures, define subsets

$$\begin{aligned} \overline{\mathbf{H}}_{\mathfrak{g}} &:= \overline{\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})} \subset \mathrm{Hilb}_{2m+1}(Z_{\mathfrak{g}}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}}), \\ \mathbf{H}_{\mathfrak{g}} &:= \overline{\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})} \subset \mathrm{Hilb}^{sn}_{2m+1}(Z_{\mathfrak{g}}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}}), \\ \mathbf{C}_{\mathfrak{g}} &:= \overline{\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})} \subset \mathrm{Chow}_{1,2}(Z_{\mathfrak{g}}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}}), \\ \mathbf{CoC}_{\mathfrak{g}} &:= \overline{\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})} \subset \mathbf{CoC}(Z_{\mathfrak{g}}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}}). \end{aligned}$$

where

$$\check{\alpha}_{\mathfrak{g}} := \begin{cases} \alpha_1^{\vee} & \text{if } \mathfrak{g} = C_r,\ r \geq 2, \\ \alpha_1^{\vee} + \alpha_r^{\vee} & \text{if } \mathfrak{g} = A_r,\ r \geq 2, \\ 2\alpha_{j_0}^{\vee} & \text{otherwise.} \end{cases}$$

By Theorem 2.2.6, these are irreducible subsets. Indeed, by Corollary 2.2.5 and Theorem 2.2.3, each of them is an irreducible component of the (semi-normalized) Hilbert scheme, the Chow scheme, and the space of complete conics, respectively. From now on, we consider $\overline{\mathbf{H}}_{\mathfrak{g}}$, $\mathbf{H}_{\mathfrak{g}}$, $\mathbf{C}_{\mathfrak{g}}$ and $\mathbf{CoC}_{\mathfrak{g}}$ as projective varieties equipped with the reduced scheme structures. Then they admit the natural G-actions, and G-equivariant birational morphisms

$$\begin{array}{ccccc} \mathbf{CoC}_{\mathfrak{g}}^{nor} & \xrightarrow{CH^{nor}} & \mathbf{H}_{\mathfrak{g}}^{nor} & \xrightarrow{FC^{nor}} & \mathbf{C}_{\mathfrak{g}}^{nor} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{CoC}_{\mathfrak{g}}^{sn} & \xrightarrow{CH^{sn}} & \mathbf{H}_{\mathfrak{g}} & \xrightarrow{FC} & \mathbf{C}_{\mathfrak{g}} \\ \downarrow & & \downarrow & & \\ \mathbf{CoC}_{\mathfrak{g}} & \xrightarrow{CH} & \overline{\mathbf{H}}_{\mathfrak{g}} & & \end{array}$$

which are isomorphisms over the loci of smooth conics ($=\mathbf{R}_{\check{\alpha}_\mathfrak{g}}$) and reducible conics.

Remark 3.1.1. There is a commutative diagram

$$\begin{array}{ccccc} \mathbf{H}_\mathfrak{g}^{nor} & \longrightarrow & \mathbf{H}_\mathfrak{g}^{sn} & \longrightarrow & \mathbf{H}_\mathfrak{g} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathbf{H}}_\mathfrak{g}^{nor} & \longrightarrow & \overline{\mathbf{H}}_\mathfrak{g}^{sn} & \longrightarrow & \overline{\mathbf{H}}_\mathfrak{g}. \end{array}$$

Every arrow represents a finite birational morphism, and so in particular, the leftmost arrow $\mathbf{H}_\mathfrak{g}^{nor} \to \overline{\mathbf{H}}_\mathfrak{g}^{nor}$ is an isomorphism. Moreover, all arrows but $\mathbf{H}_\mathfrak{g}^{nor} \to \mathbf{H}_\mathfrak{g}^{sn}$ and $\overline{\mathbf{H}}_\mathfrak{g}^{nor} \to \overline{\mathbf{H}}_\mathfrak{g}^{sn}$ are bijective, and in particular $\mathbf{H}_\mathfrak{g}^{sn} \to \overline{\mathbf{H}}_\mathfrak{g}^{sn}$ is an isomorphism. However, in general, irreducible components of a semi-normal scheme are not necessarily semi-normal (see [21, p. 308]), so it is not clear whether the morphism $\mathbf{H}_\mathfrak{g}^{sn} \to \mathbf{H}_\mathfrak{g}$ is an isomorphism.

Since $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}$ is not compact (unless $\mathfrak{g}$ is of type C; see Subsection 3.1.1), the four compactifications parametrize singular objects in their boundaries. Nonetheless, recall that their scheme structures are easy to describe (Proposition 2.2.8).

Definition 3.1.2. Let C be a conic on $Z_\mathfrak{g} \subset \mathbb{P}(\mathfrak{g})$. We say that C is *planar* if the unique plane containing it in $\mathbb{P}(\mathfrak{g})$ is also contained in $Z_\mathfrak{g}$. Otherwise C is called *non-planar*.

Remark 3.1.3. Let $C \subset Z_\mathfrak{g}$ be a conic.

1. C is a smooth conic, a reducible conic, or a double line by Proposition 2.2.8.
2. If C is a reducible conic, then C admits a smoothing in $Z_\mathfrak{g}$ (see for example [20, Theorem II.7.6]). That is, every reducible conic on $Z_\mathfrak{g}$ is a member of our four compactifications.
3. If C is planar, then we shall show that its G-conjugacy class is determined by the G-conjugacy class of the plane spanned by C. See Corollary 3.5.4.
4. By Theorem 2.2.3, the restriction of the Hilbert-Chow morphism FC

$$\mathbf{H}_\mathfrak{g} \setminus \{\text{double lines}\} \to \mathbf{C}_\mathfrak{g} \setminus \{\text{double lines}\}$$

is an isomorphism. In Subsection 3.1.1–3.1.2, we show that if $\mathfrak{g}$ is of type A or C, then there is no double line in $\mathbf{H}_\mathfrak{g}$, hence $FC : \mathbf{H}_\mathfrak{g} \to \mathbf{C}_\mathfrak{g}$ is an isomorphism.

Let us introduce two more types of smooth conics, using the G-invariant contact distribution $D \subset TZ_\mathfrak{g}$ on the adjoint variety.

Definition 3.1.4. Let C be a smooth conic on $Z_\mathfrak{g}$.

1. C is called a *twistor conic* if $T_xC \not\subset D_x$ for every $x \in C$.
2. C is called a *contact conic* if $T_xC \subset D_x$ for every $x \in C$.

It is well-known that every smooth conic is either a twistor conic or a contact conic. Indeed, if $f : \mathbb{P}^1 \to C \subset Z_\mathfrak{g}$ is a smooth conic, then since $TZ_\mathfrak{g}/D \simeq \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_\mathfrak{g}}$ (Section 2.1), we have a bundle morphism

$$\mathcal{O}_{\mathbb{P}^1}(2) \simeq T\mathbb{P}^1 \xrightarrow{df} f^*(TZ_\mathfrak{g}) \to f^*(TZ_\mathfrak{g}/D) \simeq \mathcal{O}_{\mathbb{P}^1}(2)$$

which is either an isomorphism or the zero map. In the former case, C is a twistor conic, and in the latter case, C is a contact conic.

Example 3.1.5. 1. Let C_ρ be the intersection of $Z_{\mathfrak{g}}$ and a plane

$$\mathbb{P}(E_\rho, H_\rho, E_{-\rho}) := \{[a\cdot E_\rho + b\cdot H_\rho + c\cdot E_{-\rho}] \in \mathbb{P}(\mathfrak{g}) : [a:b:c] \in \mathbb{P}^2\}.$$

Then C_ρ is a smooth conic parametrized by $\overline{\exp(\mathfrak{g}_{-\rho})\cdot o}$ where $o := [E_\rho] \in Z_{\mathfrak{g}}$, hence it is a twistor conic. Indeed, this conic is a fiber of the twistor fibration constructed in [44]. In particular, $\mathbf{R}_2(Z_{\mathfrak{g}})$ is non-empty for all $\mathfrak{g}$.

2. Every smooth planar conic is a contact conic, since every line on $Z_{\mathfrak{g}}$ is tangent to D (Subsection 2.2.2). Note that if $\mathfrak{g} = G_2$, this fact does not provide any example of contact conics since there is no plane on Z_{G_2} ([23]). In fact, in Theorem 5.2.4, we shall show that there is no contact conic on Z_{G_2}.

3.1.1 The Case of Symplectic Lie Algebras

Geometry of conics on $Z_{\mathfrak{g}}$ is particularly simple when $\mathfrak{g}$ is of type C. To see this, let $\mathfrak{g} = C_r$ for some $r \geq 2$. Then $\mathfrak{g} = \mathfrak{sp}(V)$ and $G = Sp(V)$ for a symplectic vector space V of dimension $2r$. In this case, the adjoint representation $\mathfrak{g}$ is isomorphic to $\mathrm{Sym}^2 V$ as a G-representation, and the morphism

$$\nu : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^2 V), \quad [v] \mapsto [v^2]$$

defines a G-equivariant embedding, hence $Z_{\mathfrak{g}}$ is the second Veronese embedding $\nu_2(\mathbb{P}(V))(\simeq \mathbb{P}^{2r-1})$. Thus conics on $\nu_2(\mathbb{P}(V))$ are exactly lines on $\mathbb{P}(V)$, and

$$\mathbf{R}_2(Z_{\mathfrak{g}}) \simeq \mathbf{H}_{\mathfrak{g}} \simeq \mathbf{C}_{\mathfrak{g}} \simeq \mathbf{CoC}_{\mathfrak{g}} \simeq \mathrm{Gr}(2, V).$$

Observe that $\dim \mathrm{Gr}(2, V) = 4r-4$, and the isotropic Grassmannian $\mathrm{IG}(2, V)$ is a unique closed G-orbit in $\mathrm{Gr}(2, V)$. In fact, its complement $\mathrm{Gr}(2, V) \setminus \mathrm{IG}(2, V)$ is a single G-orbit, isomorphic to a homogeneous symmetric variety $Sp_{2r}/Sp_2 \times Sp_{2r-2}$, since a non-isotropic 2-subspace is necessarily non-degenerate. In our terminology, for a conic $C \subset \nu_2(\mathbb{P}(V))$, $[C] \in \mathrm{IG}(2, V)$ if and only if C is a contact conic, and $[C] \notin \mathrm{IG}(2, V)$ if and only if C is a twistor conic. Also, every conic is non-planar, since $\nu_2(\mathbb{P}(V))$ does not contain a linear subspace.

3.1.2 The Case of Special Linear Lie Algebras

In this subsection, we describe compactifications of $\mathbf{R}_{2\alpha_i^\vee}(Z_{\mathfrak{g}})$ for $\mathfrak{g} = A_r$ ($r \geq 2$) and $i = 1, r$, i.e. the components of $\mathbf{R}_2(Z_{\mathfrak{g}})$ different from $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$. Let $\mathfrak{g} = \mathfrak{sl}(V)$ and $G = SL(V)$ where V is an $(r+1)$-dimensional vector space. If we put $V_1 := V$ and $V_r := V^*$, then the adjoint representation $\mathfrak{g}$ is a subrepresentation of $\mathfrak{gl}(V) \simeq V_1 \otimes V_r$. Moreover, for the partial flag variety $\mathrm{Fl}_{1,r}(V) := \{([x], [l]) \in \mathbb{P}(V_1) \times \mathbb{P}(V_r) : l(x) = 0\}$ and the Segre embedding

$$\sigma : \mathbb{P}(V_1) \times \mathbb{P}(V_r) \hookrightarrow \mathbb{P}(V_1 \otimes V_r) \simeq \mathbb{P}(\mathfrak{gl}(V)), \quad ([x], [l]) \mapsto [x \otimes l],$$

one can show that $\sigma(\mathrm{Fl}_{1,r}(V)) \subset \mathbb{P}(\mathfrak{sl}(V)) = \mathbb{P}(\mathfrak{g})$. Since σ is G-equivariant, σ defines an isomorphism $Z_{\mathfrak{g}} \simeq \mathrm{Fl}_{1,r}(V)$ satisfying $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}} \simeq \mathcal{O}_{\mathbb{P}(V_1 \otimes V_r)}(1)|_{\mathrm{Fl}_{1,r}(V)}$.

Consider the natural projection $p_i : \mathrm{Fl}_{1,r}(V) \to \mathbb{P}(V_i)$ for each $i = 1, r$. Then we have $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_{\mathfrak{g}}} \simeq p_1^*\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes p_r^*\mathcal{O}_{\mathbb{P}(V_r)}(1)$. Thus for any conic C on $\mathrm{Fl}_{1,r}(V)$, one of the following holds:

1. C is of degree 2 with respect to $p_1^*\mathcal{O}_{\mathbb{P}(V_1)}(1)$. In this case, we say that C is a *(2, 0)-conic*.

2. C is of degree 2 with respect to $p_r^*\mathcal{O}_{\mathbb{P}(V_r)}(1)$. In this case, we say that C is a *(0, 2)-conic.*

3. C of degree 1 with respect to both $p_1^*\mathcal{O}_{\mathbb{P}(V_1)}(1)$ and $p_r^*\mathcal{O}_{\mathbb{P}(V_r)}(1)$. In this case, we say that C is a *(1, 1)-conic.*

Observe that a smooth (2, 0)-conic is contracted by p_r. Moreover, since V_1 is the first fundamental representation of G, $\mathbf{R}_{2\alpha_1^\vee}(Z_\mathfrak{g})$ parametrizes smooth (2, 0)-conics. Similarly, a smooth (0, 2)-conic is contracted by p_1, and $\mathbf{R}_{2\alpha_r^\vee}(Z_\mathfrak{g})$ parametrizes smooth (0, 2)-conics. Furthermore, a double line cannot be a (1, 1)-conic, hence every member of $\mathbf{H}_\mathfrak{g}$ is either a smooth conic or a reducible conic. As $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g})$ parametrizes smooth (1, 1)-conics, we have

$$FC : \mathbf{H}_\mathfrak{g} \xrightarrow{\simeq} \mathbf{C}_\mathfrak{g}, \quad \text{and} \quad CH^{sn} : \mathbf{CoC}_\mathfrak{g}^{sn} \xrightarrow{\simeq} \mathbf{H}_\mathfrak{g}^{sn}.$$

The contact structure of $\mathrm{Fl}_{1,r}(V)$ can be described in terms of the projections p_1 and p_r. For each point $([x_0], [l_0]) \in \mathrm{Fl}_{1,r}(V)$, we have

$$(p_1)^{-1}([x_0]) = [x_0] \times \{[l] \in \mathbb{P}(V_r) : l(x_0) = 0\}, \quad (p_r)^{-1}([l_0]) = \{[x] \in \mathbb{P}(V_1) : l_0(x) = 0\} \times [l_0],$$

and both of them are linear $\mathbb{P}^{r-1}$ in $\mathbb{P}(V_1 \otimes V_r)$. Thus the tangent spaces of $(p_1)^{-1}([x_0])$ and $(p_r)^{-1}([l_0])$ generate a $(2r-2)$-dimensional subspace in $T_{([x_0],[l_0])}\mathrm{Fl}_{1,n+1}(V)$, which is invariant under the action of $\mathrm{Stab}_G([x_0], [l_0])$, hence it is the contact hyperplane. In particular, smooth (2, 0)- or (0, 2)-conics exist only when $r \geq 3$, and in this case, they are planar and contact conics.

To study conics contracted by p_i for $i = 1, r$, assume that $r \geq 3$, and define

$$\begin{aligned} \mathbf{H}_i &:= \overline{\mathbf{R}_{2\alpha_i^\vee}} \subset \mathrm{Hilb}^{sn}_{2m+1}(Z_\mathfrak{g}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_\mathfrak{g}}), \\ \mathbf{C}_i &:= \overline{\mathbf{R}_{2\alpha_i^\vee}} \subset \mathrm{Chow}_{1,2}(Z_\mathfrak{g}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_\mathfrak{g}}), \\ \mathbf{CoC}_i &:= \overline{\mathbf{R}_{2\alpha_i^\vee}} \subset \mathbf{CoC}(Z_\mathfrak{g}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_\mathfrak{g}}), \\ \overline{\mathbf{H}}_i &:= \overline{\mathbf{R}_{2\alpha_i^\vee}} \subset \mathrm{Hilb}_{2m+1}(Z_\mathfrak{g}, \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_{Z_\mathfrak{g}}). \end{aligned}$$

As before, these are projective G-varieties. Moreover, since planes spanned by smooth (2, 0)-conics and by smooth (0, 2)-conics are contracted by p_r and by p_1, respectively, we have a diagram of G-equivariant morphisms

$$\begin{array}{ccccc} & & \mathbf{H}_i & \xrightarrow{FC} & \mathbf{C}_i \\ & & \downarrow & & \\ \mathbf{CoC}_i & \xrightarrow{CH} & \overline{\mathbf{H}}_i & \longrightarrow & \mathbf{P}_i \end{array}$$

where $\mathbf{P}_i$ is the space of planes on $\mathrm{Fl}_{1,r}(V)$ contracted by $p_{i'}$ for $i' \in \{1, r\} \setminus \{i\}$. In fact, by [23, Theorem 4.9], $\mathbf{P}_i$ is a rational homogeneous space under the natural G-action, and

$$\mathbf{P}_1 \simeq \mathrm{Fl}_{3,r}(V), \quad \mathbf{P}_r \simeq \mathrm{Fl}_{1,r-2}(V).$$

(In particular, $\dim \mathbf{P}_i = 4r-9$.) Thus for a plane $[\mathbb{P}W_i] \in \mathbf{P}_i$ ($W_i \leq V_1 \otimes V_r$, $\dim W_i = 3$) and its stabilizer $Q_i := \mathrm{Stab}_G(W_i)$, Q_i acts on the plane $\mathbb{P}W_i$ transitively, hence there are bijective morphisms

$$G \times_{Q_i} \mathbb{P}(\mathrm{Sym}^2 W_i^*) \to \overline{\mathbf{H}}_i, \quad G \times_{Q_i} \mathbf{CoC}(\mathbb{P}W_i) \to \mathbf{CoC}_i,$$

which induce isomorphisms

$$G \times_{Q_i} \mathbb{P}(\mathrm{Sym}^2 W_i^*) \simeq \overline{\mathbf{H}}_i^{nor}, \quad G \times_{Q_i} \mathbf{CoC}(\mathbb{P}W_i) \simeq \mathbf{CoC}_i^{nor}.$$

Indeed, Q_i acts on W_i via a surjection $Q_i \to GL(W_i)$, which induces a spherical action of the Levi subgroup of Q_i on $\mathbb{P}(\mathrm{Sym}^2 W_i^*)$ and $\mathbf{CoC}(\mathbb{P}(W_i))$. In other words, the homogeneous fiber bundles $G \times_{Q_i} \mathbb{P}(\mathrm{Sym}^2 W_i^*)$ and $G \times_{Q_i} \mathbf{CoC}(\mathbb{P}W_i)$ are *parabolic inductions* of $\mathbf{P}_i$, in the sense of [41, Definition 5.9 and §20.6]. By [41, Proposition 5.10] they are G-spherical varieties of dimension $4r-4$ and of rank 2. Therefore by Proposition 2.3.7, $\overline{\mathbf{H}}_i$ has exactly three orbits, consisting of smooth conics, reducible conics and double lines, respectively, while $\mathbf{CoC}_i$ has exactly four orbits, corresponding to the Q_i-orbits in $\mathbf{CoC}(\mathbb{P}W_i)$ (see Subsection 2.2.3). As $\overline{\mathbf{H}}_i^{nor}$ is smooth, the morphism $\mathbf{H}_i^{nor} \to \overline{\mathbf{H}}_i^{nor}$ is an isomorphism. It means that the normalization $\mathbf{C}_i^{nor}$ is a G-spherical variety consisting of three G-orbits, consisting of smooth conics, reducible conics and double lines, respectively.

The previous discussion shows that the compactifications of $\mathbf{R}_{2\alpha_i^\vee}$ are related via the following morphisms

$$\begin{array}{ccc} \mathbf{CoC}_i^{nor}(\simeq G \times_{Q_i} \mathbf{CoC}(\mathbb{P}W_i)) \xrightarrow{CH^{nor}} \overline{\mathbf{H}}_i^{nor} \simeq & \mathbf{H}_i^{nor}(\simeq G \times_{Q_i} \mathbb{P}(\mathrm{Sym}^2 W_i^*)) & \xrightarrow{FC^{nor}} \mathbf{C}_i^{nor} \\ & \downarrow & \\ & \mathbf{P}_i(\simeq G/Q_i) & \end{array}$$

where each horizontal arrow is birational. Furthermore, $\mathbf{CoC}_i^{nor}$, $\mathbf{H}_i^{nor}$ and $\mathbf{C}_i^{nor}$ have unique closed G-orbits, isomorphic to

$$\mathrm{Fl}_{1,2,3,r}(V), \quad \mathrm{Fl}_{2,3,r}(V), \quad \text{and} \quad \mathrm{Fl}_{2,r}(V),$$

respectively when $i = 1$, and isomorphic to

$$\mathrm{Fl}_{1,r-2,r-1,r}(V), \quad \mathrm{Fl}_{1,r-2,r-1}(V), \quad \text{and} \quad \mathrm{Fl}_{1,r-1}(V),$$

respectively when $i = r$. (For example, for $\mathbf{H}_i^{nor}$ and $\mathbf{C}_i^{nor}$, the unique closed G-orbits are the loci of double lines.) In particular, the morphism $CH^{nor} : \mathbf{H}_i^{nor} \to \mathbf{C}_i^{nor}$ is an isomorphism if and only if $r = 3$.

Remark 3.1.6. Alternatively, $\mathbf{H}_i^{nor}$ can be described as follows. Recall that for $i' \in \{1, r\} \setminus \{i\}$, smooth conics parametrized by $\mathbf{R}_{2\alpha_i^\vee}(Z_\mathfrak{g})$ are contracted by $p_{i'}$. Consider the (dualized) Euler sequence over $\mathbb{P}(V_{i'})$

$$0 \to \mathcal{E}_{i'} \to V_i \otimes \mathcal{O}_{\mathbb{P}(V_{i'})}(-1) \to \mathcal{O}_{\mathbb{P}(V_{i'})} \to 0.$$

Here, $\mathcal{O}_{\mathbb{P}(V_{i'})}(-1)$ is the tautological line bundle over $\mathbb{P}(V_{i'})$, the quotient map is given by the natural pairing (which makes sense since $V_i \simeq V_{i'}^*$), and $\mathcal{E}_{i'}$ is a vector bundle of rank r defined as the kernel. By the above description of the fibers of $p_{i'}$, we see that $\mathrm{Fl}_{1,n+1}(V) \simeq \mathbb{P}(\mathcal{E}_{i'})$ as projective bundles over $\mathbb{P}(V_{i'})$ via a G-equivariant isomorphism. Consider the Grassmannian bundle $\mathrm{Gr}_{\mathbb{P}(V_{i'})}(3, \mathcal{E}_{i'})$ over $\mathbb{P}(V_{i'})$, equipped with the universal subbundle $\mathcal{S}_{i'}$. That is, for each $x \in \mathbb{P}(V_{i'})$, the fiber of $\mathrm{Gr}_{\mathbb{P}(V_{i'})}(3, \mathcal{E}_{i'})$ at x is the usual Grassmannian $\mathrm{Gr}(3, \mathcal{E}_{i',x})$ and the restriction $\mathcal{S}_{i'}|_x$ is the universal bundle of subspaces associated to $\mathrm{Gr}(3, \mathcal{E}_{i',x})$. Since $\mathrm{Gr}_{\mathbb{P}(V_{i'})}(3, \mathcal{E}_{i'})$ parametrizes 2-planes on the fibers of $p_{i'}$, the projective bundle $\mathbb{G}_{i'} := \mathbb{P}(\mathrm{Sym}^2 \mathcal{S}_{i'}^*)$ over $\mathrm{Gr}_{\mathbb{P}(V_{i'})}(3, \mathcal{E}_{i'})$ parametrizes conics contained in the fibers of $p_{i'}$. Indeed, $\mathbb{G}_{i'}$ is isomorphic to the normalization $\mathbf{H}_i^{nor}$ (as $\mathbb{G}_{i'}$ is smooth and the natural map $\mathbb{G}_{i'} \to \mathbf{H}_i$ is bijective).

Now we have a description of the space of (2, 0)-conics and (0, 2)-conics. For (1, 1)-conics, in Proposition 5.1.1, we shall give a blowing-up construction using our main theorem (Theorem 3.2.2).

3.2 Main Theorems

Now we state our main theorems. Recall that n is the integer defined by $2n + 1 = \dim Z_\mathfrak{g} (> 1)$.

Theorem 3.2.1. *Twistor conics form an open G-orbit in $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}$, isomorphic to a 4n-dimensional homogeneous symmetric variety $O_{\mathfrak{g}} := G/G^{\sigma}$ for some involution $\sigma : G \to G$. The Satake diagram of σ is given in Table 3.1.*

In the second column of Table 3.1, we denote by D_1 the 1-dimensional toral Lie algebra ($\simeq \mathfrak{so}_2$). In particular, G^{σ} is semi-simple if and only if $\mathfrak{g} \neq A_r$ for all $r \geq 2$.

Theorem 3.2.2. *The normalizations $\mathbf{H}_{\mathfrak{g}}^{nor}$, $\mathbf{C}_{\mathfrak{g}}^{nor}$ and $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ are projective G-symmetric varieties equipped with the G-equivariant birational morphisms*

$$\mathbf{CoC}_{\mathfrak{g}}^{nor} \xrightarrow{CH^{nor}} \mathbf{H}_{\mathfrak{g}}^{nor} \xrightarrow{FC^{nor}} \mathbf{C}_{\mathfrak{g}}^{nor}.$$

Moreover, as $O_{\mathfrak{g}}$-embeddings, their colored fans are given as follows.

1. *$\mathbf{C}_{\mathfrak{g}}^{nor}$ is a simple $O_{\mathfrak{g}}$-embedding with its colored cone in Table 3.2.*
2. *The colored fan of $\mathbf{H}_{\mathfrak{g}}^{nor}$ consists of colored cones listed in Table 3.3 and their colored faces.*
3. *The colored fan of $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ consists of colored cones listed in Table 3.4 and their colored faces.*

The proofs of Theorem 3.2.1 and Theorem 3.2.2 are given in Section 3.3 and Chapter 4, respectively.

Tables 3.2–3.4 show that $\mathbf{H}_{\mathfrak{g}}^{nor}$ (and $\mathbf{CoC}_{\mathfrak{g}}^{nor}$) is simple as a G-spherical variety if and only if $\mathfrak{g}$ is of type A, type C or an exceptional type. Otherwise, the number of closed orbits is 3 if $\mathfrak{g} = D_4$, and 2 if $\mathfrak{g} = \mathfrak{so}_N$ for $N \geq 7$ and $N \neq 8$.

Let us explain the notation in Tables 3.1–3.3, assuming Theorem 3.2.1. As in Subsection 2.3.2, let T' be a maximally σ-split torus, and B' be a Borel subgroup containing T' such that for every positive root α' with respect to B' satisfying $\overline{\alpha'} \neq 0$, we have $\sigma(\alpha') < 0$. Then there is $g \in G$ such that $T' = g \cdot T \cdot g^{-1}$ and $B' = g \cdot B \cdot g^{-1}$. (To see this, choose any $g_0 \in G$ such that $T' = g_0 \cdot T \cdot g_0^{-1}$. Then there is $w \in N(T')$ such that $B' = w \cdot (g_0 \cdot B \cdot g_0^{-1}) \cdot w^{-1}$ since $N(T')$ acts transitively on the set of Borel subgroups containing T'. So we may put $g := w \cdot g_0$.) Then we define the ingredients in Subsection 2.3.2, using T' and B'. For example, the root system R' and its simple roots S' are given by

$$R' = R \circ Ad_{g^{-1}} = \left\{\alpha' := \alpha \circ Ad_{g^{-1}} : \alpha \in R\right\}, \quad S' = S \circ Ad_{g^{-1}} = \left\{\alpha_i' := \alpha_i \circ Ad_{g^{-1}} : \alpha_i \in S\right\}.$$

We index restricted simple roots $S'_{O_{\mathfrak{g}}} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ as in the last column of Table 3.1. When $R'_{O_{\mathfrak{g}}}$ is reduced, the indexing agrees with [32, Reference Chapter, Table 1]. For B'-colors of $O_{\mathfrak{g}}$, i.e. elements in $\mathcal{D}(O_{\mathfrak{g}})$, we use the following notation:

- If $\mathfrak{g} = A_r$ ($r \geq 2$), then $O_{\mathfrak{g}}$ is indeed isomorphic to G'/K' given in Remark 2.3.12. For $S'_{O_{\mathfrak{g}}}$ and $\mathcal{D}(O_{\mathfrak{g}})$, we keep the notation of Remark 2.3.12.
- If $\mathfrak{g} \neq A_r$ ($r \geq 2$), then by Theorem 3.2.1, $K = G^{\sigma}$ is semi-simple, hence the map $\epsilon : \mathcal{D}(O_{\mathfrak{g}}) \to \mathcal{E}$ is bijective onto $\frac{1}{2} \cdot (S'_{O_{\mathfrak{g}}})^{\vee}$ by Theorem 2.3.11. So we put $\mathcal{D}_i := \epsilon^{-1}(\lambda_i^{\vee}/2) \in \mathcal{D}(O_{\mathfrak{g}})$.

In this notation, the (positive) Weyl chamber is given by $-\mathcal{V} = \mathbb{Q}_{\geq 0}\langle \gamma_1, \ldots \gamma_m \rangle$ where γ_j's are defined by the relations $\langle \lambda_i, \gamma_j \rangle = \delta_{ij}$. If $\mathfrak{g}$ is of type A or C, then since the rank of $R'_{O_{\mathfrak{g}}}$ is at most 2, $\{\gamma_j\}$ can be easily computed. In other cases, since $R'_{O_{\mathfrak{g}}}$ is reduced, the expression of γ_j in terms of the restricted simple coroots $(S'_{O_{\mathfrak{g}}})^{\vee} = \{\lambda_1^{\vee}, \ldots, \lambda_m^{\vee}\}$ can be read off from the jth rows of the matrices in [32, Reference Chapter, Table 2]. In fact, the ith column $(c_{1i} \cdots c_{mi})^t$ of the matrix for $R'_{O_{\mathfrak{g}}}$ means

$$\pi_i = \sum_{k=1}^{m} c_{ki} \cdot \lambda_k$$

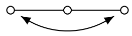
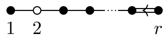
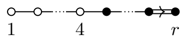
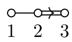
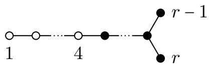
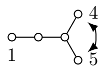
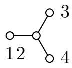
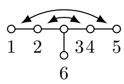
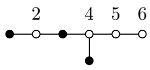
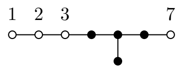
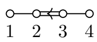
$\mathfrak{g}$	$\mathfrak{g}^\sigma$(=Lie algebra of G^σ)	Satake Diagram of $O_\mathfrak{g}$	$R'_{O_\mathfrak{g}}$	$S'_{O_\mathfrak{g}}$
A_r ($r \geq 4$)	$A_{r-2} \oplus A_1 \oplus D_1$		BC_2	$\lambda_1 = \overline{\alpha'_1} = \overline{\alpha'_r}$ $\lambda_2 = \overline{\alpha'_2} = \overline{\alpha'_{r-1}}$
A_3	$A_1 \oplus A_1 \oplus D_1$		C_2	$\lambda_1 = \overline{\alpha'_1} = \overline{\alpha'_3}$ $\lambda_2 = \overline{\alpha'_2}$
A_2	$A_1 \oplus D_1$		BC_1	$\lambda = \overline{\alpha'_1} = \overline{\alpha'_2}$
C_r ($r \geq 3$)	$C_{r-1} \oplus A_1$		BC_1	$\lambda = \overline{\alpha'_2}$
C_2	$A_1 \oplus A_1$		A_1	$\lambda = \overline{\alpha'_2}$
B_r ($r \geq 4$)	$B_{r-2} \oplus A_1 \oplus A_1$		B_4	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 4)$
B_3	$A_1 \oplus A_1 \oplus A_1$		B_3	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 3)$
D_r ($r \geq 6$)	$D_{r-2} \oplus A_1 \oplus A_1$		B_4	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 4)$
D_5	$D_3 \oplus A_1 \oplus A_1$		B_4	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 3)$ $\lambda_4 = \overline{\alpha'_4} = \overline{\alpha'_5}$
D_4	$A_1 \oplus A_1 \oplus A_1 \oplus A_1$		D_4	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 4)$
E_6	$A_5 \oplus A_1$		F_4	$\lambda_1 = \overline{\alpha'_1} = \overline{\alpha'_5}$ $\lambda_2 = \overline{\alpha'_2} = \overline{\alpha'_4}$ $\lambda_3 = \overline{\alpha'_3}$ $\lambda_4 = \overline{\alpha'_6}$
E_7	$D_6 \oplus A_1$		F_4	$\lambda_1 = \overline{\alpha'_2}$ $\lambda_2 = \overline{\alpha'_4}$ $\lambda_3 = \overline{\alpha'_5}$ $\lambda_4 = \overline{\alpha'_6}$
E_8	$E_7 \oplus A_1$		F_4	$\lambda_1 = \overline{\alpha'_7}$ $\lambda_2 = \overline{\alpha'_3}$ $\lambda_3 = \overline{\alpha'_2}$ $\lambda_4 = \overline{\alpha'_1}$
F_4	$C_3 \oplus A_1$		F_4	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 4)$
G_2	$A_1 \oplus A_1$		G_2	$\lambda_i = \overline{\alpha'_i}$ $(1 \leq i \leq 2)$

Table 3.1: Satake diagram and the restricted root system of $O_\mathfrak{g}$.

$\mathfrak{g}$	$(\mathcal{C}(\mathbf{C}_{\mathfrak{g}}^{nor}), \mathcal{F}(\mathbf{C}_{\mathfrak{g}}^{nor}))$
A_r $(r \geq 3)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2 \rangle, \varnothing)$
A_2, C_r $(r \geq 2)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma \rangle, \varnothing)$
B_r $(r \geq 4)$, D_r $(r \geq 5)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, \lambda_2^\vee, \lambda_4^\vee \rangle, \{\mathcal{D}_2, \mathcal{D}_4\})$
B_3	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_3, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$
D_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_3, -\gamma_4, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$
E_r $(r = 6, 7, 8)$, F_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4, \lambda_1^\vee, \lambda_2^\vee, \lambda_4^\vee \rangle, \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_4\})$
G_2	$(\mathbb{Q}_{\geq 0}\langle -\gamma_2, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$

Table 3.2: Colored cone of $\mathbf{C}_{\mathfrak{g}}^{nor}$ in $\mathbb{Q}\langle (R'_{O_{\mathfrak{g}}})^\vee \rangle$.

$\mathfrak{g}$	Maximal Colored Cones in $\mathfrak{F}(\mathbf{H}_{\mathfrak{g}}^{nor})$
A_r $(r \geq 3)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2 \rangle, \varnothing)$
A_2, C_r $(r \geq 2)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma \rangle, \varnothing)$
B_r $(r \geq 4)$, D_r $(r \geq 5)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, \lambda_2^\vee, \lambda_4^\vee \rangle, \{\mathcal{D}_2, \mathcal{D}_4\})$, $(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$
B_3	$(\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_2, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$ for $i = 1, 3$
D_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_j, \lambda_2^\vee : j \in \{1, 2, 3, 4\} \smallsetminus \{i\} \rangle, \{\mathcal{D}_2\})$ for $i = 1, 3, 4$
E_r $(r = 6, 7, 8)$, F_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4, \lambda_1^\vee, \lambda_4^\vee \rangle, \{\mathcal{D}_1, \mathcal{D}_4\})$
G_2	$(\mathbb{Q}_{\geq 0}\langle -\gamma_2, \lambda_2^\vee \rangle, \{\mathcal{D}_2\})$

Table 3.3: Colored fan of $\mathbf{H}_{\mathfrak{g}}^{nor}$ in $\mathbb{Q}\langle (R'_{O_{\mathfrak{g}}})^\vee \rangle$.

$\mathfrak{g}$	Maximal Colored Cones in $\mathfrak{F}(\mathbf{CoC}_{\mathfrak{g}}^{nor})$
A_r $(r \geq 3)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2 \rangle, \varnothing)$
A_2, C_r $(r \geq 2)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma \rangle, \varnothing)$
B_r $(r \geq 4)$, D_r $(r \geq 5)$	$(\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3, \lambda_4^\vee \rangle, \{\mathcal{D}_4\})$, $(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3 \rangle, \varnothing)$
B_3	$(\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_2, -\gamma_1 - \gamma_3 \rangle, \varnothing)$ for $i = 1, 3$
D_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_j, -\gamma_1 - \gamma_3 - \gamma_4 : j \in \{1, 2, 3, 4\} \smallsetminus \{i\} \rangle, \varnothing)$ for $i = 1, 3, 4$
E_r $(r = 6, 7, 8)$, F_4	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_3, -\gamma_4, \lambda_1^\vee \rangle, \{\mathcal{D}_1\})$
G_2	$(\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2 \rangle, \varnothing)$

Table 3.4: Colored fan of $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ in $\mathbb{Q}\langle (R'_{O_{\mathfrak{g}}})^\vee \rangle$.

where π_i is the ith fundamental weight of $R'_{O_\mathfrak{g}}$, hence if $\gamma_j = \sum_{l=1}^m d_{lj} \cdot \lambda_l^\vee$, then

$$d_{lj} = \langle \pi_l, \gamma_j \rangle = c_{jl},$$

i.e. the coefficients $c_{j1}, \ldots, c_{jm}$ of γ_j form the jth row. We summarize the result as follows.

- If $\mathfrak{g} = A_r$ $(r \geq 4)$, then $R'_{O_\mathfrak{g}} = BC_2$ and

$$\gamma_1 = \lambda_1^\vee + \lambda_2^\vee,$$
$$\gamma_2 = \lambda_1^\vee + 2\lambda_2^\vee$$

  where $\lambda_1^\vee (= \lambda_1^V)$ and $\lambda_2^\vee (= \lambda_2^V/2)$ are basis elements of $(R'_{O_\mathfrak{g}})^\vee$ (see Remark 2.3.12).

- If $\mathfrak{g} = A_3$, then $R'_{O_\mathfrak{g}} = C_2$ and

$$\gamma_1 = \lambda_1^\vee + \lambda_2^\vee,$$
$$\gamma_2 = \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee.$$

- If $\mathfrak{g} = A_2$ or C_r $(r \geq 2)$, then $R'_{O_\mathfrak{g}} = BC_1$, and $\gamma = \gamma_1 = \lambda^\vee$ where $\lambda^\vee (= \lambda^V/2)$ means the basis element of $(R'_{O_\mathfrak{g}})^\vee$ (hence $\langle \lambda, \lambda^\vee \rangle = 1$; see Remark 2.3.12).

- If $\mathfrak{g} = B_{r\geq 4}$ or $D_{r\geq 5}$, then $R'_{O_\mathfrak{g}} = B_4$ and

$$\gamma_1 \coloneqq \lambda_1^\vee + \lambda_2^\vee + \lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_2 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + 2\lambda_3^\vee + \lambda_4^\vee,$$
$$\gamma_3 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + \frac{3}{2}\lambda_4^\vee,$$
$$\gamma_4 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + 2\lambda_4^\vee.$$

- If $\mathfrak{g} = B_3$, then $R'_{O_\mathfrak{g}} = B_3$ and

$$\gamma_1 \coloneqq \lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee,$$
$$\gamma_2 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + \lambda_3^\vee,$$
$$\gamma_3 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + \frac{3}{2}\lambda_3^\vee.$$

- If $\mathfrak{g} = D_4$, then $R'_{O_\mathfrak{g}} = D_4$ and

$$\gamma_1 \coloneqq \lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_2 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + \lambda_3^\vee + \lambda_4^\vee,$$
$$\gamma_3 \coloneqq \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee + \lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_4 \coloneqq \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee + \lambda_4^\vee.$$

- If $\mathfrak{g}$ is of an exceptional type other than G_2, then $R'_{O_\mathfrak{g}} = F_4$ and

$$\gamma_1 \coloneqq 2\lambda_1^\vee + 3\lambda_2^\vee + 4\lambda_3^\vee + 2\lambda_4^\vee,$$
$$\gamma_2 \coloneqq 3\lambda_1^\vee + 6\lambda_2^\vee + 8\lambda_3^\vee + 4\lambda_4^\vee,$$
$$\gamma_3 \coloneqq 2\lambda_1^\vee + 4\lambda_2^\vee + 6\lambda_3^\vee + 3\lambda_4^\vee,$$
$$\gamma_4 \coloneqq \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + 2\lambda_4^\vee.$$

- If $\mathfrak{g} = G_2$, then $R'_{O_\mathfrak{g}} = G_2$ and

$$\gamma_1 := 2\lambda_1^\vee + 3\lambda_2^\vee, \quad \gamma_2 := \lambda_1^\vee + 2\lambda_2^\vee.$$

Let us close this section after collecting colored faces of the colored cones in Table 3.2 and Table 3.3. Since each colored face $(\mathcal{C}, \mathcal{F})$ is determined by its underlying cone $\mathcal{C}$, it suffices to write $\mathcal{C}$. In the following list, we classify the colored faces according to their dimensions.

1. The nonzero non-maximal elements in the colored fan defined by Table 3.2:
   (a) When $\mathfrak{g} = A_r$ $(r \geq 3)$:
      i. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_1\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2\rangle$.
   (b) When $\mathfrak{g} = A_2$ or C_r $(r \geq 2)$, there is no nonzero proper colored face.
   (c) When $\mathfrak{g}$ is $B_{r\geq 4}$ or $D_{r\geq 5}$:
      i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, \lambda_4^\vee\rangle$.
      ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 2, 4\}$, $\mathbb{Q}_{\geq 0}\langle -\gamma_4, \lambda_4^\vee\rangle$.
      iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 4$.
   (d) When $\mathfrak{g} = B_3$:
      i. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_3\rangle$.
      ii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 3$.
   (e) When $\mathfrak{g} = D_4$:
      i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 3, 4\}$.
      ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 2, 3, 4\}$.
      iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 3, 4$.
   (f) When $\mathfrak{g}$ is one of E_6, E_7, E_8 and F_4:
      i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4, \lambda_1^\vee\rangle$.
      ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_1, \lambda_1^\vee\rangle$.
      iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 4$.
   (g) When $\mathfrak{g} = G_2$:
      i. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_2\rangle$.
2. The nonzero non-maximal elements in the colored fan defined by Table 3.3:
   (a) When $\mathfrak{g} = A_r$ $(r \geq 3)$:
      i. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_1\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2\rangle$.
   (b) When $\mathfrak{g} = A_2$ or C_r $(r \geq 2)$, there is no nonzero proper colored face.
   (c) When $\mathfrak{g}$ is $B_{r\geq 4}$ or $D_{r\geq 5}$:
      i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, \lambda_4^\vee\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_i, \lambda_2^\vee\rangle$ for $i \in \{1, 4\}$.
      ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 2, 4\}$, $\mathbb{Q}_{\geq 0}\langle -\gamma_k, \lambda_k^\vee\rangle$ for $k \in \{2, 4\}$.
      iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 4$.
   (d) When $\mathfrak{g} = B_3$:

i. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_3\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, \lambda_2^\vee\rangle$.

ii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 3$.

(e) When $\mathfrak{g} = D_4$:

i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 3, 4\}$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_k, \lambda_2^\vee\rangle$ for $k \in \{1, 3, 4\}$.

ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_j\rangle$ for $i \neq j \in \{1, 2, 3, 4\}$, $\mathbb{Q}_{\geq 0}\langle -\gamma_2, \lambda_2^\vee\rangle$.

iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 2, 3, 4$.

(f) When $\mathfrak{g}$ is one of E_6, E_7, E_8 and F_4:

i. dim = 3: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4, \lambda_1^\vee\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4, \lambda_4^\vee\rangle$.

ii. dim = 2: $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_4\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_1, \lambda_1^\vee\rangle$, $\mathbb{Q}_{\geq 0}\langle -\gamma_4, \lambda_4^\vee\rangle$.

iii. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_i\rangle$ for $i = 1, 4$.

(g) When $\mathfrak{g} = G_2$:

i. dim = 1: $\mathbb{Q}_{\geq 0}\langle -\gamma_2\rangle$.

3.3 Sphericality of Space of Twistor Conics

In this section, we prove Theorem 3.2.1, and that in every tangent direction off the contact distribution, there is exactly one twistor conic.

Lemma 3.3.1 ([17, Lemma 5])**.** *The unipotent radical $R^u(P)$ of the isotropy group P at $o \in Z_\mathfrak{g}$ acts transitively on the open subset $\mathbb{P}(T_oZ_\mathfrak{g}) \setminus \mathbb{P}(D_o)$ in the projectivized tangent space.*

Proof. This statement is shown in the proof of [17, Lemma 5], in the case where $\mathfrak{g}$ is not of type A or C. In fact, its proof works for all $Z_\mathfrak{g}$. For the sake of completeness, let us record the proof for all $Z_\mathfrak{g}$.

Recall the contact gradation (see Section 2.1)

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathfrak{p} = \mathfrak{g}^0, \quad \mathfrak{g}_i = \bigoplus_{\langle\alpha\,|\,\rho\rangle = i} \mathfrak{g}_i, \quad \forall i \neq 0.$$

Observe that if $\alpha \in R$ with $\langle\alpha\,|\,\rho\rangle = \pm 1$, then $\alpha \mp \rho$ is also a root such that $\langle\alpha \mp \rho\,|\,\rho\rangle = \mp 1$. Thus $[\mathfrak{g}_{-\rho}, \mathfrak{g}_1] = \mathfrak{g}_{-1}$. Since

$$T_oZ_\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{p}, \quad D_o \simeq \mathfrak{g}^{-1}/\mathfrak{p},$$

we see that $[\mathfrak{g}_{-\rho}] \in \mathbb{P}(T_oZ_\mathfrak{g}) \setminus \mathbb{P}(D_o)$, and its orbit under the $R^u(P)(= \exp(\mathfrak{g}^1))$-action is open in $\mathbb{P}(T_oZ_\mathfrak{g}) \setminus \mathbb{P}(D_o)$. Furthermore, since $\mathbb{P}(D_o)$ is a hyperplane, $\mathbb{P}(T_oZ_\mathfrak{g}) \setminus \mathbb{P}(D_o)$ is an affine space ($\simeq \mathbb{C}^{2n}$), hence every $R^u(P)$-orbit is closed by [5, Proposition 4.10]. Therefore $R^u(P)\cdot[\mathfrak{g}_{-\rho}] = \mathbb{P}(T_oZ_\mathfrak{g}) \setminus \mathbb{P}(D_o)$. □

Lemma 3.3.2. *1. The normal bundle of a twistor conic in $Z_\mathfrak{g}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2n}$.*

2. $\dim \mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g}) = 4n$*.*

3. The locus of twistor conics is an open G-orbit in $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g})$.

Proof. Recall that $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g})$ is an open subscheme of the Hilbert scheme (Corollary 2.2.5), containing the locus of twistor conics (Subsection 3.1.2). Let C be a twistor conic and $f : \mathbb{P}^1 \to C \subset Z_\mathfrak{g}$ an embedding. By Lemma 3.3.1 and [20, Theorem II.3.11], f is free over $0 \mapsto f(0)$, i.e.

$$f^*TZ_\mathfrak{g} \simeq \bigoplus_{i=1}^{2n+1} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad \text{for some } a_1 \geq \cdots \geq a_{2n+1} > 0.$$

Since the anti-canonical bundle $K_{Z_{\mathfrak{g}}}^{-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(n+1)|_{Z_{\mathfrak{g}}}$ (see [24]) and C is a conic, we have

$$2(n+1) = \deg_{\mathbb{P}^1} f^* K_{Z_{\mathfrak{g}}}^{-1} = \deg_{\mathbb{P}^1} f^* T Z_{\mathfrak{g}} = \sum_{i=1}^{2n+1} a_i.$$

This is possible only if $a_1 = 2$ and $a_2 = \cdots = a_{2n+1} = 1$, hence the normal bundle of C is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2n}$. In particular, the dimension of the Hilbert scheme at $[C]$ is $4n$, and so $\dim \mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}}) = 4n$.

Now consider the space $\mathrm{Hom}_{bir}(\mathbb{P}^1, Z_{\mathfrak{g}})$ of morphisms from $\mathbb{P}^1$ to $Z_{\mathfrak{g}}$ which are birational onto their images. Let V be the closure of the $G \times \mathrm{Aut}(\mathbb{P}^1)$-orbit containing $[f]$ in $\mathrm{Hom}_{bir}(\mathbb{P}^1, Z_{\mathfrak{g}})$. By Lemma 3.3.1, for arbitrary $x \in Z_{\mathfrak{g}}$, $\mathrm{Locus}(V, 0 \mapsto x)$ is open in $Z_{\mathfrak{g}}$. Thus by the proof of [20, Proposition IV.2.5], for general points x and y in $Z_{\mathfrak{g}}$, we have

$$\begin{aligned}\dim V &= \dim\{[h] \in V : h(0) = x,\, h(\infty) = y\} + \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V, 0 \mapsto x)\\ &\geq 4n + 3.\end{aligned}$$

Therefore by [20, Theorem II.2.15], the G-orbit containing $[C]$ in the Chow scheme is at least $4n$-dimensional, hence each orbit containing a twistor conic is (Zariski) open in $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$. Since $\mathbf{R}_{\check{\alpha}_{\mathfrak{g}}}(Z_{\mathfrak{g}})$ is irreducible (Theorem 2.2.6), all twistor conics are in the same G-orbit. ☐

Lemma 3.3.3. *The stabilizer $\mathrm{Stab}_G(C_\rho)$ of the twistor conic $C_\rho = Z_{\mathfrak{g}} \cap \mathbb{P}(E_\rho, H_\rho, E_{-\rho})$ introduced in Example 3.1.5 is the connected Lie subgroup K of G associated to the Lie subalgebra*

$$\mathfrak{k} := \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}.$$

In particular, $\mathrm{Stab}_G(C_\rho)$ is a reductive subgroup of same rank with G, and the Dynkin diagram of its semi-simple part can be obtained by deleting the nodes adjacent to $-\rho$ in the extended Dynkin diagram of $\mathfrak{g}$ (Table 2.1).

Proof. Observe that $g \in G$ stabilizes C_ρ if and only if it stabilizes $\mathbb{P}(E_\rho, H_\rho, E_{-\rho})$. Thus the Lie algebra $\mathfrak{k}$ is contained in the Lie algebra of $\mathrm{Stab}_G(C_\rho)$, hence $K \subset \mathrm{Stab}_G(C_\rho)$. For the converse, let $g \in \mathrm{Stab}_G(C_\rho)$, and then claim that $g \in K$. Observe that the $\mathfrak{sl}_2$ algebra $\mathbb{C} \cdot H_\rho \oplus \mathfrak{g}_\rho \oplus \mathfrak{g}_{-\rho}$ is contained in $\mathfrak{k}$, and the corresponding SL_2 acts transitively on C_ρ. Thus we may assume that g fixes $o \in C_\rho$, i.e. $g \in P$. Now consider the Levi decomposition $P = R^u(P) \rtimes L$ where $R^u(P)$ is the unipotent radical of P and L is the standard Levi subgroup. That is, the Lie algebras of $R^u(P)$ and L are given by $\mathfrak{g}^1$ and by $\mathfrak{g}_0$, respectively. Since $L \subset K$, we may assume that $g \in R^u(P)$, say $g = \exp(X)$ for some $X \in \mathfrak{g}^1$. Note that since $g = \exp(X)$ is unipotent, $\exp(tX) \in \mathrm{Stab}_G(C_\rho) \cap P$ for every $t \in \mathbb{C}$, hence $\exp(tX)$ stabilizes $T_o C_\rho \simeq \mathfrak{g}_{-\rho} \mod \mathfrak{p}$ in $T_o Z_{\mathfrak{g}} \simeq \mathfrak{g}/\mathfrak{p}$. Therefore

$$[X, \mathfrak{g}_{-\rho}] \mod \mathfrak{p} \quad \subset \quad \mathfrak{g}_{-\rho} \mod \mathfrak{p} \quad \text{in } \mathfrak{g}/\mathfrak{p}.$$

This is possible only if $X \in \mathfrak{g}_\rho$, hence $g \in K$. ☐

Theorem 3.3.4. *Let v be a nonzero tangent vector of $Z_{\mathfrak{g}}$ which does not belong to the contact distribution D. Then there is exactly one twistor conic tangent to v.*

Proof. By Lemma 3.3.1, we may assume that $v \in T_o C_\rho$. Suppose that there is a twistor conic C tangent to v at o. By Lemma 3.3.2, there is $g \in G$ such that $C = g \cdot C_\rho$. We claim that g is indeed contained in $K = \mathrm{Stab}_G(C_\rho)$. Since K contains the Lie subgroup corresponding to $\mathbb{C} \cdot H_\rho \oplus \mathfrak{g}_\rho \oplus \mathfrak{g}_{-\rho}$, we may assume that g fixes o, i.e. $g \in P$. Since the tangent directions of C and C_ρ coincide, g stabilizes $T_o C_\rho$, hence by repeating the argument in the proof of Lemma 3.3.3, we conclude that $g \in K$. ☐

Proposition 3.3.5. *There is an involution $\sigma : G \to G$ such that the fixed point subgroup G^σ is K.*

Proof. By the proof of [44, Theorem 5.4], there is an inner involution on a real form of $\mathfrak{g}$ such that the $(+1)$-eigenspace is a real form of $\mathfrak{k}$. By taking its $\mathbb{C}$-linear extension over $\mathfrak{g}$, since G is simply connected, we obtain a holomorphic involution $\sigma : G \to G$, and the $(+1)$-eigenspace of its differential at the identity element is $\mathfrak{k}$. Since G^σ is connected ([39, Theorem 8.1]), we have $G^\sigma = K$. ☐

Proof of Theorem 3.2.1. By Lemma 3.3.2, Lemma 3.3.3 and Proposition 3.3.5, twistor conics form an open orbit in $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g})$, isomorphic to the homogeneous symmetric variety $O_\mathfrak{g} := G/G^\sigma$ of dimension $4n$. From Lemma 3.3.3 and the classification of Satake diagrams for simple Lie algebras in [36, Table 1] (see also [41, Table 26.3]), we easily obtain the Satake diagram of $O_\mathfrak{g}$ for each $\mathfrak{g}$. (Note that SO_4 is of type $A_1 \times A_1$ and SO_3 is of type A_1.) ☐

Remark 3.3.6. If $\mathfrak{g}$ is either A_2 or of type C, then by Theorem 3.2.1, the reduced root system is of rank 1 and the image of the color map is not contained in the valuation cone $\mathcal{V}$. Thus there is a unique G-equivariant completion of $O_\mathfrak{g}$, which is associated to the colored cone $(\mathcal{V}, \varnothing)$. In particular, Theorem 3.2.2 for $\mathfrak{g} = A_2, C_r$ $(r \geq 2)$ follows.

3.4 Tangent Directions of Contact Conics

Next, we study geometry of contact conics. Namely, we find an equation satisfied by tangent vectors of contact conics. Using this, we show that when $\mathfrak{g}$ is not of type C, there is no smooth conic in a general direction of D, while tangent directions of twistor conics dominate $\mathbb{P}(T_o Z_\mathfrak{g}) \setminus \mathbb{P}(D_o)$ by Theorem 3.3.4. (Observe that if $\mathfrak{g}$ is of type C, then $Z_\mathfrak{g} = \nu_2(\mathbb{P}^{2n+1})$ and there is a smooth conic in every direction. See Subsection 3.1.1.)

In this section, every argument is based on Lie theoretic computation, and independent of spherical geometry. For the sake of simplicity, we choose root vectors $\{E_\alpha \in \mathfrak{g}_\alpha\}$ of $\mathfrak{g}$ as in [14, Theorem 5.5, Ch. III]. Namely, our root vectors satisfy

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad \forall \alpha \in R$$

and

$$(N_{\alpha,\beta})^2 = \frac{q(1-p)}{2} \cdot \langle \alpha, \alpha \rangle \quad \forall \alpha, \beta \in R \text{ satisfying } \alpha + \beta \in R$$

where

$$p := \min\{m \in \mathbb{Z} : \beta + m\alpha \in R\} \quad \text{and} \quad q := \max\{m \in \mathbb{Z} : \beta + m\alpha \in R\}.$$

Proposition 3.4.1. *For nonzero $v \in D_o$, there is a line or a smooth conic tangent to v if and only if v satisfies*

$$[v, [v, [v, E_\rho]]] = 0$$

after identifying v with an element in $\mathfrak{g}$ via the vector space isomorphism $D_o \simeq \mathfrak{g}_{-1}$.

Proof. Note that $[v, E_\rho] \neq 0$ in $\mathfrak{g}$ whenever $v \in D_o \setminus \{0\}$. If $[v, [v, [v, E_\rho]]] = 0$, then

$$\exp(t \cdot v) \cdot o = \left[E_\rho + t \cdot [v, E_\rho] + \frac{t^2}{2} \cdot [v, [v, E_\rho]]\right] \in \mathbb{P}(\mathfrak{g}), \quad \forall t \in \mathbb{C}.$$

It parametrizes a line if $[v, [v, E_\rho]] = 0$. If $[v, [v, E_\rho]] \neq 0$, then since $E_\rho (\in \mathfrak{g}_2)$, $[v, E_\rho] (\in \mathfrak{g}_1)$ and $[v, [v, E_\rho]] (\in \mathfrak{g}_0)$ are linearly independent, it parametrizes a smooth conic.

Conversely, assume that there is a line or smooth conic C in $Z_{\mathfrak{g}}$ such that $o \in C$ and $v \in T_oC$. Suppose that C is a line. For $\alpha \in N(\rho)$, it can be easily seen that $[E_{-\alpha}, [E_{-\alpha}, E_\rho]] = 0$, hence there exists a line in direction $[\mathfrak{g}_{-\alpha}] \in \mathbb{P}(D_o)$. By Theorem 2.2.7, there exists $p \in L$ such that $Ad_p(v) \in \mathfrak{g}_{-\alpha}$ for some $\alpha \in N(\rho)$ and the standard Levi subgroup L of P (i.e. $T_eL = \mathfrak{g}_0$). Thus

$$Ad_p([v, [v, E_\rho]]) \in [\mathfrak{g}_{-\alpha}, [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\rho]] = 0.$$

Thus we may assume that C is a conic. Then since the exponential map defines a local isomorphism near the origin

$$T_oZ_{\mathfrak{g}} \simeq \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \to Z_{\mathfrak{g}}, \quad X \mapsto \exp(X) \cdot o,$$

there is a holomorphic map $F : t \mapsto F(t) \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ such that $F(0) = 0$, $F'(0) = v$ and

$$\exp(F(t)) \cdot o, \quad \forall t \text{ near } 0 \in \mathbb{C}$$

is a local parametrization of C near o. For all t near $0 \in \mathbb{C}$,

$$\exp(F(t)) \cdot o = \left[E_\rho + \sum_{k=1}^{\infty} \frac{1}{k!} (ad_{F(t)})^k (E_\rho) \right].$$

Since $F(t) \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$,

$$(ad_{F(t)})^k(E_\rho) \in \begin{cases} \mathbb{C} \cdot H_\rho \oplus \mathfrak{g}_1 & (\text{if } k = 1); \\ \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 & (\text{if } k = 2); \\ \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} & (\text{if } k = 3); \\ \mathfrak{g}_{-\rho} & (\text{if } k = 4); \\ 0 & (\text{if } k \geq 5). \end{cases}$$

Therefore in the affine chart $E_\rho + (\mathfrak{t} \oplus \bigoplus_{\alpha \neq \rho} \mathfrak{g}_\alpha) \simeq \mathfrak{t} \oplus \bigoplus_{\alpha \neq \rho} \mathfrak{g}_\alpha$ of $\mathbb{P}(\mathfrak{g})$, $\exp(F(t)) \cdot o$ is given by

$$\sum_{k=1}^{4} \frac{1}{k!} (ad_{F(t)})^k (E_\rho).$$

For sufficiently small t, we have the Taylor expansion of F

$$F(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} F^{(i)}, \quad F^{(i)} \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \quad F^{(1)} := v,$$

and

$$\begin{aligned}
&\sum_{k=1}^{4} \frac{1}{k!} (ad_{F(t)})^k (E_\rho) \\
&= \sum_{k=1}^{4} \sum_{i_1, \dots, i_k \geq 1} \frac{t^{i_1 + \cdots + i_k}}{k! \cdot i_1! \cdots i_k!} [F^{(i_1)}, \cdots [F^{(i_{k-1})}, [F^{(i_k)}, E_\rho]] \cdots] \\
&= t \cdot [F^{(1)}, E_\rho] \\
&\quad + t^2 \cdot \left(\frac{1}{2} [F^{(2)}, E_\rho] + \frac{1}{2} [F^{(1)}, [F^{(1)}, E_\rho]] \right) \\
&\quad + t^3 \cdot \left(\frac{1}{6} [F^{(3)}, E_\rho] + \frac{1}{2 \cdot 2} ([F^{(1)}, [F^{(2)}, E_\rho]] + [F^{(2)}, [F^{(1)}, E_\rho]]) + \frac{1}{6} [F^{(1)}, [F^{(1)}, [F^{(1)}, E_\rho]]] \right) \\
&\quad + t^4 \cdot \left(\frac{1}{24} [F^{(4)}, E_\rho] + \frac{1}{2} (\frac{1}{6} [F^{(1)}, [F^{(3)}, E_\rho]] + \frac{1}{4} [F^{(2)}, [F^{(2)}, E_\rho]] + \frac{1}{6} [F^{(3)}, [F^{(1)}, E_\rho]]) \right. \\
&\qquad + \frac{1}{6 \cdot 2} ([F^{(1)}, [F^{(1)}, [F^{(2)}, E_\rho]]] + [F^{(1)}, [F^{(2)}, [F^{(1)}, E_\rho]]] + [F^{(2)}, [F^{(1)}, [F^{(1)}, E_\rho]]]) \\
&\qquad \left. + \frac{1}{24} [F^{(1)}, [F^{(1)}, [F^{(1)}, [F^{(1)}, E_\rho]]]] \right) \\
&\quad + O(t^5).
\end{aligned}$$

The plane spanned by C in $\mathbb{P}(\mathfrak{g})$ can be written as $\mathbb{P}(\mathfrak{g}_\rho \oplus V)$ for some 2-dimensional subspace $V \le \mathfrak{t} \oplus \bigoplus_{\alpha \neq \rho} \mathfrak{g}_\alpha$. Then the intersection of the plane spanned by C and the affine open subset $E_\rho + (\mathfrak{t} \oplus \bigoplus_{\alpha \neq \rho} \mathfrak{g}_\alpha)$ is $E_\rho + V$, and so all derivatives of $\sum_{k=1}^4 \frac{1}{k!}(ad_{F(t)})^k(E_\rho)$ at $t = 0$ are elements of V. Consider the first and second derivatives

$$[F^{(1)}, E_\rho], \quad [F^{(2)}, E_\rho] + [F^{(1)}, [F^{(1)}, E_\rho]].$$

Assume that the three vectors

$$E_\rho, \quad [F^{(1)}, E_\rho], \quad [F^{(2)}, E_\rho] + [F^{(1)}, [F^{(1)}, E_\rho]]$$

are linearly dependent in $\mathfrak{g}$. Since

$$[F^{(1)}, E_\rho] = [v, E_\rho] \in \mathfrak{g}_1 \setminus \{0\},$$

$$[F^{(1)}, [F^{(1)}, E_\rho]] \in \mathfrak{g}_0,$$

and

$$[F^{(2)}, E_\rho] \in \mathbb{C} \cdot H_\rho \oplus \mathfrak{g}_1,$$

$[F^{(1)}, E_\rho]$ and $[F^{(2)}, E_\rho] + [F^{(1)}, [F^{(1)}, E_\rho]]$ are linearly dependent, hence

$$[v, [v, E_\rho]] = [F^{(1)}, [F^{(1)}, E_\rho]] \in \mathbb{C} \cdot H_\rho.$$

Let us write $[v, [v, E_\rho]] = c \cdot H_\rho$ for some $c \in \mathbb{C}$. Note that for every $\alpha \in R$ with $\langle \alpha | \rho \rangle = -1$, $[H_\rho, E_\alpha] = \langle \alpha, \rho \rangle \cdot E_\alpha = -\frac{\langle \rho, \rho \rangle}{2} \cdot E_\alpha$, hence

$$[H_\rho, v] = -\frac{\langle \rho, \rho \rangle}{2} \cdot v. \tag{3.1}$$

By the invariance of the Killing form under the adjoint representation,

$$c \cdot \langle H_\rho, H_\rho \rangle = \langle H_\rho, [v, [v, E_\rho]] \rangle = \langle [[H_\rho, v], v], E_\rho \rangle = 0,$$

hence $c = 0$ and $[v, [v, E_\rho]] = 0$.

Thus we may assume that the three vectors are linearly independent. That is, the plane spanned by C is

$$\mathbb{P}(E_\rho, [F^{(1)}, E_\rho], [F^{(2)}, E_\rho] + [F^{(1)}, [F^{(1)}, E_\rho]])$$

and its intersection with the affine open subset $E_\rho + (\mathfrak{t} \oplus \bigoplus_{\alpha \neq \rho} \mathfrak{g}_\alpha)$ is identified with

$$V = \mathbb{C}\langle [F^{(1)}, E_\rho], [F^{(2)}, E_\rho] + [F^{(1)}, [F^{(1)}, E_\rho]] \rangle.$$

Now for each $i \ge 1$, write $F^{(i)}$ as

$$F^{(i)} = X^{(i)} + x^{(i)} \cdot E_{-\rho}, \quad X^{(i)} \in \mathfrak{g}_{-1}, \quad x^{(i)} \in \mathbb{C}.$$

Note that $x^{(1)} = 0$ and $X^{(1)} = v$. Then the coefficient of t^3 in the above formula, which is proportional to the third derivative at $t = 0$, is

$$\begin{aligned}
&\frac{1}{6}[F^{(3)}, E_\rho] + \frac{1}{4}([F^{(1)}, [F^{(2)}, E_\rho]] + [F^{(2)}, [F^{(1)}, E_\rho]]) + \frac{1}{6}[F^{(1)}, [F^{(1)}, [F^{(1)}, E_\rho]]] \\
&= \underbrace{\frac{1}{6}[X^{(3)}, E_\rho]}_{\in \mathfrak{g}_1} \\
&\quad \underbrace{+ \frac{1}{6}(-x^{(3)})H_\rho + \frac{1}{4}\left([v, [X^{(2)}, E_\rho]] + [X^{(2)}, [v, E_\rho]]\right)}_{\in \mathfrak{g}_0} \\
&\quad \underbrace{+ \frac{1}{4}x^{(2)}[H_\rho, v] + \frac{1}{4}x^{(2)}[E_{-\rho}, [v, E_\rho]] + \frac{1}{6}[v, [v, [v, E_\rho]]]}_{\in \mathfrak{g}_{-1}}.
\end{aligned}$$

Since it is contained in the vector space V, spanned by $[v, E_\rho]$ and $-x^{(2)}H_\rho + [X^{(2)}, E_\rho] + [v, [v, E_\rho]]$, the $\mathfrak{g}_{-1}$-component is zero:

$$\frac{1}{4}x^{(2)}[H_\rho, v] + \frac{1}{4}x^{(2)}[E_{-\rho}, [v, E_\rho]] + \frac{1}{6}[v, [v, [v, E_\rho]]] = 0.$$

By the Jacobi identity,

$$\begin{aligned}[E_{-\rho}, [v, E_\rho]] &= [v, [E_{-\rho}, E_\rho]] \\ &= [H_\rho, v] \\ &= -\frac{\langle\rho, \rho\rangle}{2} \cdot v, \quad (\because \text{Equation (3.1)})\end{aligned}$$

hence

$$[v, [v, [v, E_\rho]]] = \frac{3}{2}\langle\rho, \rho\rangle x^{(2)} \cdot v.$$

In particular,

$$[v, [v, [v, [v, E_\rho]]]] = 0.$$

So the coefficient of t^4 in the above formula is

$$\begin{aligned}&\underbrace{\frac{1}{24}[X^{(4)}, E_\rho]}_{\in\mathfrak{g}_1} \\ &\underbrace{+\frac{1}{24}(-x^{(4)})H_\rho + \frac{1}{12}[v, [X^{(3)}, E_\rho]] + \frac{1}{8}[X^{(2)}, [X^{(2)}, E_\rho]] + \frac{1}{12}[X^{(3)}, [v, E_\rho]]}_{\in\mathfrak{g}_0} \\ &\underbrace{\begin{aligned}&+\frac{1}{12}x^{(3)}[H_\rho, v] + \frac{1}{8}x^{(2)}([X^{(2)}, [E_{-\rho}, E_\rho]] + [E_{-\rho}, [X^{(2)}, E_\rho]]) + \frac{1}{12}x^{(3)}[E_{-\rho}, [v, E_\rho]] \\ &\quad+\frac{1}{12}\left([v, [v, [X^{(2)}, E_\rho]]] + [v, [X^{(2)}, [v, E_\rho]]] + [X^{(2)}, [v, [v, E_\rho]]]\right)\end{aligned}}_{\in\mathfrak{g}_{-1}} \\ &\underbrace{+\frac{1}{8}(x^{(2)})^2(-\langle\rho, \rho\rangle)E_{-\rho} + \frac{1}{12}x^{(2)}\left([v, [v, [E_{-\rho}, E_\rho]]] + [v, [E_{-\rho}, [v, E_\rho]]] + [E_{-\rho}, [v, [v, E_\rho]]]\right)}_{\in\mathfrak{g}_{-2}(=\mathfrak{g}_{-\rho})}.\end{aligned}$$

It is also a linear combination of $[v, E_\rho]$ and $-x^{(2)}H_\rho + [X^{(2)}, E_\rho] + [v, [v, E_\rho]]$, hence the $\mathfrak{g}_{-\rho}$-component is zero:

$$\frac{1}{8}(x^{(2)})^2(-\langle\rho, \rho\rangle)E_{-\rho} + \frac{1}{12}x^{(2)}\left([v, [v, [E_{-\rho}, E_\rho]]] + [v, [E_{-\rho}, [v, E_\rho]]] + [E_{-\rho}, [v, [v, E_\rho]]]\right) = 0.$$

By the Jacobi identity,

$$\begin{aligned}[E_{-\rho}, [v, [v, E_\rho]]] &= [v, [E_{-\rho}, [v, E_\rho]]] \\ &= [v, [v, [E_{-\rho}, E_\rho]]] \\ &= 0, \quad (\because \text{Equation (3.1)}).\end{aligned}$$

Therefore $x^{(2)} = 0$, which means that

$$[v, [v, [v, E_\rho]]] = \frac{3}{2}\langle\rho, \rho\rangle x^{(2)} \cdot v = 0.$$

□

Corollary 3.4.2. *If $\mathfrak{g} \neq C_r$, $r \geq 2$, then there is no smooth conic in a general direction of D.*

Proof. By Proposition 3.4.1, it is enough to show that there is nonzero $v \in \mathfrak{g}_{-1}$ such that $[v, [v, [v, E_\rho]]] \neq 0$. Since $\mathfrak{g}$ is not of type C, there is a long root α satisfying $\langle \alpha | \rho \rangle = \langle \rho | \alpha \rangle = -1$ (for example, elements of $N(\rho)$ times -1). Then $\rho + \alpha \in R$ but $\rho + 2\alpha \notin R$ and $\rho + 2(-\alpha - \rho) \notin R$. Also, $\alpha + \rho$ is a long root. Moreover, by our choice of root vectors,

$$(N_{\alpha, \rho})^2 = (N_{-\alpha-\rho, \rho})^2 = \frac{1}{2}\langle \rho, \rho \rangle.$$

Now a straightforward computation shows that $E_\alpha + E_{-\alpha-\rho}$ does not satisfy the equation. □

Remark 3.4.3. Proposition 3.4.1 can be used to show that every smooth conic in the G_2-adjoint variety Z_{G_2} is a twistor conic. Indeed, if $\mathfrak{g} = G_2$, then one can prove that for $v \in \mathfrak{g}_{-1}$,

$$[v, [v, [v, E_\rho]]] = 0 \quad \text{if and only if} \quad [v, [v, E_\rho]] = 0.$$

It means that every contact conic is planar. However, Z_{G_2} does not contain a plane ([23, Section 4.3]), hence a smooth conic on Z_{G_2} cannot be tangent to D. This fact is recovered in Theorem 5.2.4, as a corollary of our main theorem (Theorem 3.2.2).

3.5 Classification of Borel Fixed Conics

In this section, we study B-fixed points of the compactifications $\mathbf{H}_\mathfrak{g}$, $\mathbf{C}_\mathfrak{g}$ and $\mathbf{CoC}_\mathfrak{g}$, which can be regarded as points corresponding to the most singular conics. Namely, we compute the isotropy groups of the closed orbits of $\mathbf{H}_\mathfrak{g}$, $\mathbf{C}_\mathfrak{g}$ and $\mathbf{CoC}_\mathfrak{g}$.

Recall that if $\mathfrak{g} = C_r$ $(r \geq 2)$, then $\mathbf{R}_{\check{\alpha}_\mathfrak{g}}(Z_\mathfrak{g})$ is compact, and it contains a unique closed orbit $\mathrm{IG}(2, \mathbb{C}^{2r})$ (Subsection 3.1.1). In other words, the space of conics contains a unique B-fixed point represented by a contact conic whose stabilizer is P_{α_2}.

Thus we mainly consider the case where $\mathfrak{g}$ is not of type C. Recall that if $\mathfrak{g}$ is of type A, then the three compactifications are all isomorphic to each other (Subsection 3.1.2), and so it is enough to consider one of them.

Lemma 3.5.1. *If $\mathfrak{g} = A_r$ $(r \geq 2)$, then $\mathbf{H}_\mathfrak{g}$ contains a unique B-fixed point, represented by a reducible conic*

$$\mathbb{P}(E_\rho, E_{\rho-\alpha_1}) \cup \mathbb{P}(E_\rho, E_{\rho-\alpha_r}).$$

Moreover, its stabilizer is $P_{\alpha_1, \alpha_2, \alpha_{r-1}, \alpha_r}$.

Proof. For simplicity, put $\mathcal{L}_i := \mathbb{P}(E_\rho, E_{\rho-\alpha_i})$ for $i = 1, r$. Then the reducible conic $\mathcal{L}_1 \cup \mathcal{L}_r$ has stabilizer $P_{\alpha_1, \alpha_2, \alpha_{r-1}, \alpha_r}$ (see Subsection 2.2.2), and in particular it is a B-stable conic.

To show the uniqueness, let C be a B-stable conic on $Z_\mathfrak{g}$ such that $[C] \in \mathbf{H}_\mathfrak{g}$. As we have seen in Subsection 3.1.2, C cannot be a double line. If C is a reducible conic, then each of its components is B-stable. However, by Theorem 2.2.7, there only two B-stable lines on $Z_\mathfrak{g}$, hence $C = \mathcal{L}_1 \cup \mathcal{L}_r$. If C is smooth, then it contains $o = [E_\rho]$ (as o is a unique B-fixed point of $Z_\mathfrak{g}$ and by the Borel fixed point theorem), and its projective tangent line at o is also B-stable. That is, $\mathcal{L}_i$ is tangent to C at o for some $i = 1, r$. This shows that for the plane $\mathcal{P}$ spanned by C, $\mathcal{P} \cap Z_\mathfrak{g}$ contains $C \cup \mathcal{L}_i$. Since $Z_\mathfrak{g}$ is defined by quadrics in $\mathbb{P}(\mathfrak{g})$, we see that $\mathcal{P} \subset Z_\mathfrak{g}$. This is a contradiction, otherwise double lines on $\mathcal{P}$ represent points in $\mathbf{H}_\mathfrak{g}$. □

Lemma 3.5.2. *Suppose that $\mathfrak{g}$ is not of type A or C, and let α_{j_0} be the unique element of $N(\rho)$.*

1. *The line* $\mathcal{L}_B := \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}})$ *is a unique* B*-stable line in* $\mathbb{P}(\mathfrak{g})$*.*

2. *Any* B*-stable plane in* $\mathbb{P}(\mathfrak{g})$ *contains the* B*-stable line* $\mathcal{L}_B$*, and is of form*

$$\mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$$

where $\beta \in S$ *is a neighbor of* α_{j_0} *in the Dynkin diagram of* $\mathfrak{g}$*.*

3. *In* $\mathbb{P}(\mathfrak{g})$*, there is no* B*-stable conic which is smooth or reducible.*

Proof. If $\mathcal{L} \subset \mathbb{P}(\mathfrak{g})$ is a B-stable line, then $\mathcal{L}$ contains $o = [E_\rho]$ which is the unique B-fixed point in $\mathbb{P}(\mathfrak{g})$. Thus $\mathcal{L} = \mathbb{P}(E_\rho, v)$ for some $v \in \mathfrak{g}$. Moreover, since α_{j_0} is the unique simple root which is not orthogonal to ρ, $\rho - \alpha_{j_0}$ is the maximum in $R \setminus \{\rho\}$. By $\mathfrak{b}$-stability of $\mathcal{L}$, we have $\mathcal{L} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}})$.

If $\mathcal{P}$ is a B-stable plane in $\mathbb{P}(\mathfrak{g})$, then B acts on the space of lines on $\mathcal{P}$, which is compact. Thus by the uniqueness of $\mathcal{L}_B$, $\mathcal{P}$ contains $\mathcal{L}_B$ and $\mathcal{P} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, v)$ for some $v \in \mathfrak{g}$. By $\mathfrak{b}$-stability, we may choose v as a root vector corresponding to a maximal element in $R \setminus \{\rho, \rho - \alpha_{j_0}\}$, which is exactly of form $\rho - \alpha_{j_0} - \beta$ for some neighbor β of α_{j_0} in the Dynkin diagram. (In fact, $\rho - 2\alpha_{j_0}$ is not a root since α_{j_0} is long hence $\langle \alpha_{j_0} | \rho \rangle = \langle \rho | \alpha_{j_0} \rangle = 1$.)

For the last statement, consider two different lines $\mathcal{L}_1$ and $\mathcal{L}_2$ such that their union $\mathcal{L}_1 \cup \mathcal{L}_2$ is B-stable. Since B is irreducible, each $\mathcal{L}_i$ should be B-stable, hence $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_B$, a contradiction. Now assume that there is a B-stable smooth conic C in $\mathbb{P}(\mathfrak{g})$. Then the plane spanned by C is also B-stable, hence there is some neighbor $\beta \in S$ of α_{j_0} such that the plane $\mathcal{P} := \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$ contains C. Moreover, $o \in C$ and the line $\mathcal{L}_B$ is tangent to C at o. Therefore in $\mathcal{P}$, C is defined by a quadratic equation

$$a_{11}x_1^2 + a_{22}x_2^2 + x_0x_2 + a_{12}x_1x_2 = 0$$

for some $a_{ij} \in \mathbb{C}$ where the homogeneous coordinate on $\mathcal{P}$ is chosen so that $[x_0 : x_1 : x_2] = [x_0E_\rho + x_1E_{\rho-\alpha_{j_0}} + x_2E_{\rho-\alpha_{j_0}-\beta}]$. Then the above equation should be B-stable up to scalar multiplication, however a simple computation shows that for each $H \in \mathfrak{t}$, $\exp(-H)$ sends the equation to

$$\begin{aligned} 0 = {} & a_{11}\left(x_1 e^{(\rho-\alpha_{j_0})(H)}\right)^2 + a_{22}\left(x_2 e^{(\rho-\alpha_{j_0}-\beta)(H)}\right)^2 \\ & + \left(x_0 e^{\rho(H)}\right)\left(x_2 e^{(\rho-\alpha_{j_0}-\beta)(H)}\right) + a_{12}\left(x_1 e^{(\rho-\alpha_{j_0})(H)}\right)\left(x_2 e^{(\rho-\alpha_{j_0}-\beta)(H)}\right). \end{aligned}$$

This implies that $a_{11} = a_{22} = a_{12} = 0$, which is impossible. ☐

It is not difficult to compute all B-stable planes in $\mathbb{P}(\mathfrak{g})$ and their stabilizers in G using Lemma 3.5.2. For example, given a B-stable plane $\mathcal{P} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$, it can be shown that its stabilizer is a parabolic subgroup P_I where $I \subset S$ is

$$(N(\alpha_{j_0}) \cup N(\beta)) \setminus \{\alpha_{j_0}, \beta\} \quad \text{if } \beta \text{ is long,}$$

and

$$(N(\alpha_{j_0}) \cup N(\beta)) \setminus \{\alpha_{j_0}\} \quad \text{if } \beta \text{ is short.}$$

(Alternatively, I is a set of $\gamma \in S$ such that $\rho - \alpha_{j_0} - \gamma \in R \setminus \{\rho - \alpha_{j_0} - \beta\}$, or $\rho - \alpha_{j_0} - \beta - \gamma \in R$.) These are listed in Table 3.5. We also indicate whether a plane is contained in $Z_\mathfrak{g}$ or not, by the following observation: A B-stable plane $\mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$ is contained in $Z_\mathfrak{g}$ if and only if β is a long root.

$\mathfrak{g}$	B-stable plane $\mathcal{P}$	$\mathrm{Stab}_G(\mathcal{P})$
B_r $(r \geq 4)$	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	P_{α_1, α_4}
B_3	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$ $(\notin Z_{\mathfrak{g}})$	P_{α_1, α_3}
D_r $(r \geq 6)$	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	P_{α_1, α_4}
D_5	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	$P_{\alpha_1, \alpha_4, \alpha_5}$
D_4	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3, α_4}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	P_{α_1, α_4}
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_4})$	P_{α_1, α_3}
E_6	$\mathbb{P}(E_\rho, E_{\rho-\alpha_6}, E_{\rho-\alpha_3-\alpha_6})$	P_{α_2, α_4}
E_7	$\mathbb{P}(E_\rho, E_{\rho-\alpha_6}, E_{\rho-\alpha_5-\alpha_6})$	P_{α_4}
E_8	$\mathbb{P}(E_\rho, E_{\rho-\alpha_1}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_3}
F_4	$\mathbb{P}(E_\rho, E_{\rho-\alpha_4}, E_{\rho-\alpha_3-\alpha_4})$	P_{α_2}
G_2	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$ $(\notin Z_{\mathfrak{g}})$	P_{α_1}

Table 3.5: B-stable planes in $\mathbb{P}(\mathfrak{g})$ and their stabilizers.

This is because, since

$$\begin{aligned}\frac{|\rho-\alpha_{j_0}-\beta|^2}{|\beta|^2} &= \frac{|\rho|^2+|\alpha_{j_0}|^2+|\beta|^2-2\langle\rho, \alpha_{j_0}\rangle-2\langle\rho, \beta\rangle+2\langle\alpha_{j_0}, \beta\rangle}{|\beta|^2} \\ &= \frac{|\alpha_{j_0}|^2}{|\beta|^2}+1+\langle\alpha_{j_0}\,|\,\beta\rangle \quad (\because\langle\alpha_{j_0}\,|\,\rho\rangle=1, \quad \langle\rho, \beta\rangle=0) \\ &= 1 \quad (\because[32, \text{Problem 8, §4.2}]),\end{aligned}$$

β is short if and only if $\rho-\alpha_{j_0}-\beta$ is short, which is equivalent to saying that $[E_{\rho-\alpha_{j_0}-\beta}] \notin Z_{\mathfrak{g}}$. Then the observation follows, since if $\mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta}) \notin Z_{\mathfrak{g}}$, then their intersection is a B-stable double line supported on $\mathcal{L}_B$ by Lemma 3.5.2.

Remark 3.5.3. The stabilizers of B-stable planes contained in $Z_{\mathfrak{g}}$ are also given in [23, Theorem 4.9]. On the other hand, comparing Table 3.5 and Theorem 2.2.7, we see that the stabilizers of B-stable planes not in $Z_{\mathfrak{g}}$ are equal to $\mathrm{Stab}_G(\mathcal{L}_B)$.

Corollary 3.5.4. *Suppose that $\mathcal{P}$ is a plane on $Z_{\mathfrak{g}}$. Then the restriction map $Stab_G(\mathcal{P}) \to Aut(\mathcal{P})$ is surjective.*

As a consequence, the G-conjugacy class of a planar conic is only depending on the G-conjugacy class of the plane spanned by it and its scheme structure. More precisely, if C_i is a planar conic on a plane $\mathcal{P}_i \subset Z_{\mathfrak{g}}$ for $i = 1, 2$, then C_1 and C_2 are G-conjugate to each other if and only if $\mathcal{P}_1$ and $\mathcal{P}_2$ are G-conjugate planes and C_1 are C_2 isomorphic as schemes.

Proof of Corollary 3.5.4. Recall that if $\mathfrak{g}$ is of type C, then $Z_{\mathfrak{g}} = \nu_2(\mathbb{P}^{2n+1})$, hence there is no plane on $Z_{\mathfrak{g}}$. If $\mathfrak{g} = A_r$, $r \geq 2$ and $\mathcal{P}$ is a plane on $Z_{\mathfrak{g}}$, then conics on $\mathcal{P}$ are either (2, 0)- or (0, 2)-conics, since a double line cannot be a (1, 1)-conic (see Subsection 3.1.2). Thus in the notation of Subsection 3.1.2, $\mathcal{P}$ is contracted by $Z_{\mathfrak{g}} \to \mathbb{P}(V_1)$ or by $Z_{\mathfrak{g}} \to \mathbb{P}(V_r)$, and so the statement follows from the discussion in Subsection 3.1.2.

Now we may assume that $\mathfrak{g}$ is not of type A or C. Since the space of linear subspaces in $Z_{\mathfrak{g}}$ is the disjoint union of rational homogeneous spaces ([23, Theorem 4.9]), we may assume that $\mathcal{P}$ is B-stable. Then by Lemma 3.5.2, we can write $\mathcal{P} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$. Observe that the Lie algebra of $\mathrm{Stab}_G(\mathcal{P})$ contains $\mathfrak{g}_{-\alpha_{j_0}} + \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha_{j_0}-\beta}$ by Table 3.5. Moreover, with respect to the homogeneous coordinate $[x : y : z] = [xE_\rho + yE_{\rho-\alpha_{j_0}} + zE_{\rho-\alpha_{j_0}-\beta}]$ on $\mathcal{P}$, $\exp(\mathfrak{g}_{-\alpha_{j_0}} + \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha_{j_0}-\beta})$ (respectively, $\exp(\mathfrak{g}_{\alpha_{j_0}} + \mathfrak{g}_{\beta} + \mathfrak{g}_{\alpha_{j_0}+\beta})$) generates all lower (respectively, upper) triangular matrices of which diagonal elements are 1 in $\mathrm{Aut}(\mathcal{P}) \simeq PGL_3$. Since the maximal torus T is sent to the group of diagonal matrices, it follows that $\mathrm{Stab}_G(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$ is surjective. □

Corollary 3.5.5. *Assume that $\mathfrak{g}$ is not of type A or C.*

1. *$\mathbf{C}_{\mathfrak{g}}$ has a unique closed G-orbit $\simeq G/Stab_G(\mathcal{L}_B)$ which is the locus of double lines.*

2. *For a closed G-orbit $\mathcal{O} \subset \mathbf{H}_{\mathfrak{g}}$, let $\mathcal{P}_{\mathcal{O}}(\subset \mathbb{P}(\mathfrak{g}))$ be the plane spanned by a B-fixed point in $\mathcal{O}$. Then the assignment $\mathcal{O} \mapsto \mathcal{P}_{\mathcal{O}}$ is a bijective map from the set of closed G-orbits in $\mathbf{H}_{\mathfrak{g}}$ to the set of B-stable planes on $\mathbb{P}(\mathfrak{g})$. Under this map, the closed orbit sent to a B-stable plane $\mathcal{P} \subset \mathbb{P}(\mathfrak{g})$ is isomorphic to $G/Stab_G(\mathcal{L}_B) \cap Stab_G(\mathcal{P})$.*

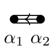
3. *For a closed G-orbit $\mathcal{O} \subset \mathbf{CoC}_{\mathfrak{g}}$, let $x \in \mathcal{O}$ be a B-fixed point and $\mathcal{P}_{\mathcal{O}}$ the plane spanned by the conic $CH(x)(\in \overline{\mathbf{H}}_{\mathfrak{g}})$. Then the assignment $\mathcal{O} \mapsto \mathcal{P}_{\mathcal{O}}$ defines a bijective map from the closed G-orbits in $\mathbf{CoC}_{\mathfrak{g}}$ to the B-stable planes in $\mathbb{P}(\mathfrak{g})$. Furthermore, if $\mathcal{P}_{\mathcal{O}} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta})$ and $Aut(\mathcal{P}_{\mathcal{O}})$ is identified with PGL_3 with respect to the ordered basis $\{E_\rho, E_{\rho-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}-\beta}\}$, then $Stab_G(x)$ is the preimage of the subgroup of upper triangular matrices under the restriction map $Stab_G(\mathcal{P}_{\mathcal{O}}) \to Aut(\mathcal{P}_{\mathcal{O}}) \simeq PGL_3$.*

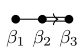
Proof. 1. Since closed orbits in $\mathbf{C}_{\mathfrak{g}}$ and $\mathbf{H}_{\mathfrak{g}}$ are projective, by Lemma 3.5.2, closed orbits must consist of double lines. Thus the first statement follows from Theorem 2.2.7.

2. For each closed orbit $\mathcal{O}$ in $\mathbf{H}_{\mathfrak{g}}$, consider its unique B-fixed point. This point is represented by a double line, say $\mathcal{L}_{\mathcal{O}}$, in $Z_{\mathfrak{g}}$ such that $(\mathcal{L}_{\mathcal{O}})^{red} = \mathcal{L}_B$ by Lemma 3.5.2. Now define $\mathcal{P}_{\mathcal{O}}$ to be the unique plane in $\mathbb{P}(\mathfrak{g})$ which contains $\mathcal{L}_{\mathcal{O}}$ as a closed subscheme. Then the map $\mathcal{O} \mapsto \mathcal{P}_{\mathcal{O}}$ is injective. Since the stabilizer of $\mathcal{L}_{\mathcal{O}}$ is equal to $\mathrm{Stab}_G(\mathcal{L}_B) \cap \mathrm{Stab}_G(\mathcal{P}_{\mathcal{O}})$, we see that $\mathcal{O} \simeq G/\mathrm{Stab}_G(\mathcal{L}_B) \cap \mathrm{Stab}_G(\mathcal{P}_{\mathcal{O}})$.

   For bijectivity, observe that the injective map $\mathcal{O} \mapsto \mathcal{P}_{\mathcal{O}}$ is surjective if there is only one B-stable plane in $\mathbb{P}(\mathfrak{g})$, which is the case when $\mathfrak{g}$ is of an exceptional type. On the other hand, if every B-stable plane in $\mathbb{P}(\mathfrak{g})$ is contained in $Z_{\mathfrak{g}}$, i.e. when $\mathfrak{g} \neq B_3, G_2$, then every B-stable double line in $\mathbb{P}(\mathfrak{g})$ represents a point in $\mathbf{H}_{\mathfrak{g}}$, hence the map $\mathcal{O} \mapsto \mathcal{P}_{\mathcal{O}}$ is surjective.

   Thus it suffices to show the surjectivity when $\mathfrak{g} = B_3$. Let us index simple roots of G_2 and B_3 so that their Dynkin diagrams are given by

   for G_2, and 

   for B_3.

   For roots $\alpha \in R_{G_2}$ and $\beta \in R_{B_3}$, we denote by $(G_2)_\alpha$ and $(B_3)_\beta$ the corresponding root spaces. Root vectors are denoted by $E_\alpha \in (G_2)_\alpha$ and $E_\beta \in (B_3)_\beta$ as before. Then there is an embedding

$G_2 \hookrightarrow B_3$ as a Lie subalgebra so that

$$\begin{cases} (G_2)_{\alpha_1}; \\ (G_2)_{\alpha_2}; \\ (G_2)_{\alpha_1+\alpha_2}; \\ (G_2)_{2\alpha_1+\alpha_2}; \\ (G_2)_{3\alpha_1+\alpha_2}; \\ (G_2)_{3\alpha_1+2\alpha_2} \end{cases} \text{are generated by} \begin{cases} E_{\beta_1} + c_{10}E_{\beta_3}; \\ E_{\beta_2}; \\ E_{\beta_1+\beta_2} + c_{12} \cdot E_{\beta_2+\beta_3}; \\ E_{\beta_2+2\beta_3} + c_{21} \cdot E_{\beta_1+\beta_2+\beta_3}; \\ E_{\beta_1+\beta_2+2\beta_3}; \\ E_{\beta_1+2\beta_2+2\beta_3}, \end{cases} \text{respectively,}$$

for some nonzero constants $c_{ij} \in \mathbb{C}^\times$. See for example [26, p. 84].

Now consider the induced embedding between adjoint varieties $Z_{G_2} \hookrightarrow Z_{B_3}$. Since the ideal of $Z_\mathfrak{g} \subset \mathbb{P}(\mathfrak{g})$ is generated by a system of quadrics, by Corollary 3.5.5 and Table 3.5, the scheme-theoretic intersection of Z_{G_2} and the plane

$$\begin{aligned} \mathcal{P} &:= \mathbb{P}(E_{3\alpha_1+2\alpha_2}, E_{3\alpha_1+\alpha_2}, E_{2\alpha_1+\alpha_2}) \text{ in } \mathbb{P}(G_2) \\ &= \mathbb{P}(E_{\beta_1+2\beta_2+2\beta_3}, E_{\beta_1+\beta_2+2\beta_3}, E_{\beta_2+2\beta_3} + c_{21} \cdot E_{\beta_1+\beta_2+\beta_3}) \text{ in } \mathbb{P}(B_3) \end{aligned}$$

is a double line which represents a point in $\mathbf{H}_{G_2}$, hence a point $[Z_{G_2} \cap_{\text{sch}} \mathcal{P}]$ in $\mathbf{H}_{B_3}$. Furthermore, the T-orbit closure of $[Z_{G_2} \cap_{\text{sch}} \mathcal{P}]$ contains two boundary points, which are conics spanning planes

$$\mathcal{P}_1 := \mathbb{P}(E_{\beta_1+2\beta_2+2\beta_3}, E_{\beta_1+\beta_2+2\beta_3}, E_{\beta_2+2\beta_3}), \quad \text{and} \quad \mathcal{P}_2 := \mathbb{P}(E_{\beta_1+2\beta_2+2\beta_3}, E_{\beta_1+\beta_2+2\beta_3}, E_{\beta_1+\beta_2+\beta_3}),$$

respectively. These are the B-stable planes for $\mathfrak{g} = B_3$ in Table 3.5. Since $\mathcal{P}_2 \not\subset Z_{B_3}$, the only conic on $\mathcal{P}_2$ which represents a point in $\mathbf{H}_{B_3}$ is the set-theoretic intersection $\mathcal{P}_2 \cap_{\text{sch}} Z_{B_3}$. It is a B-stable double line, hence $\mathcal{P}_2 = \mathcal{P}_{G\cdot[\mathcal{P}_2 \cap_{\text{sch}} Z_{B_3}]}$. On the other hand, since $\mathcal{P}_1 \subset Z_{B_3}$, $\mathcal{P}_1$ is the plane corresponding to the orbit of the B-stable double line on it.

3. Recall that the space of complete conics on $\mathbb{P}^2$ contains a unique closed PGL_3-orbit, and its isotropy group is a Borel subgroup of PGL_3. Then it suffices to observe that the subgroup of upper triangular matrices is contained in the image of the restriction map $\text{Stab}_G(\mathcal{P}_\mathcal{O}) \to PGL_3$. In fact, this observation implies that the assignment $\mathcal{O} \mapsto \mathcal{P}_\mathcal{O}$ is injective, and its surjectivity follows from the second statement.

□

Using Corollary 3.5.5, one can easily compute the isotropy groups of closed orbits. In the case of $\mathbf{C}_\mathfrak{g}$, the isotropy group of the closed orbit is $P_{N(\alpha_{j_0})}$ by Theorem 2.2.7. For $\mathbf{H}_\mathfrak{g}$ and $\mathbf{CoC}_\mathfrak{g}$, we summarize the result in Table 3.6. Here, $\mathcal{O}$ means a closed G-orbit either in $\mathbf{H}_\mathfrak{g}$ or in $\mathbf{CoC}_\mathfrak{g}$, and $\mathcal{P}_\mathcal{O}$ denotes the B-stable plane corresponding to $\mathcal{O}$ in the sense of Corollary 3.5.5. In the third and fourth columns, the isotropy groups of $\mathcal{O}$ containing B are given.

$\mathfrak{g}$	B-stable plane $\mathcal{P}_{\mathcal{O}}$	Isotropy group of $\mathcal{O} \subset \mathbf{H}_{\mathfrak{g}}$	Isotropy group of $\mathcal{O} \subset \mathbf{CoC}_{\mathfrak{g}}$
B_r $(r \geq 4)$	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_1, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3}$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	$P_{\alpha_1, \alpha_3, \alpha_4}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}$
B_3	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_1, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3} (= B)$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	P_{α_1, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3} (= B)$
D_r $(r \geq 6)$	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_1, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3}$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	$P_{\alpha_1, \alpha_3, \alpha_4}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}$
D_5	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_1, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3}$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	$P_{\alpha_1, \alpha_3, \alpha_4, \alpha_5}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5} (= B)$
D_4	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	$P_{\alpha_1, \alpha_3, \alpha_4}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} (= B)$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_3})$	$P_{\alpha_1, \alpha_3, \alpha_4}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} (= B)$
	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_2-\alpha_4})$	$P_{\alpha_1, \alpha_3, \alpha_4}$	$P_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} (= B)$
E_6	$\mathbb{P}(E_\rho, E_{\rho-\alpha_6}, E_{\rho-\alpha_3-\alpha_6})$	$P_{\alpha_2, \alpha_3, \alpha_4}$	$P_{\alpha_2, \alpha_3, \alpha_4, \alpha_6}$
E_7	$\mathbb{P}(E_\rho, E_{\rho-\alpha_6}, E_{\rho-\alpha_5-\alpha_6})$	P_{α_4, α_5}	$P_{\alpha_4, \alpha_5, \alpha_6}$
E_8	$\mathbb{P}(E_\rho, E_{\rho-\alpha_1}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_2, α_3}	$P_{\alpha_1, \alpha_2, \alpha_3}$
F_4	$\mathbb{P}(E_\rho, E_{\rho-\alpha_4}, E_{\rho-\alpha_3-\alpha_4})$	P_{α_2, α_3}	$P_{\alpha_2, \alpha_3, \alpha_4}$
G_2	$\mathbb{P}(E_\rho, E_{\rho-\alpha_2}, E_{\rho-\alpha_1-\alpha_2})$	P_{α_1}	$P_{\alpha_1, \alpha_2} (= B)$

Table 3.6: Isotropy groups of the closed G-orbits $\mathcal{O}$ in $\mathbf{H}_{\mathfrak{g}}$ and in $\mathbf{CoC}_{\mathfrak{g}}$.

Chapter 4. Colored Fans of Spaces of Conics

This whole chapter is devoted to the proof of our main theorem: Theorem 3.2.2. For the proof, we use the results of the previous Chapter 3, especially on the open orbits and the closed orbits of the compactifications $\mathbf{H}_{\mathfrak{g}}$, $\mathbf{C}_{\mathfrak{g}}$ and $\mathbf{CoC}_{\mathfrak{g}}$, to compute the colored fans of their normalizations.

To do this, recall the notation given in Section 3.2. Namely, we consider a maximally σ-split torus $T' = g\cdot T\cdot g^{-1}$ and a Borel subgroup $B' = g\cdot B\cdot g^{-1}$ such that for a positive root α with respect to B' such that $\overline{\alpha} := \alpha|_{T'_1} \neq 0$, then $\sigma(\alpha) < 0$. Also recall the expression of the generators γ_j of the Weyl chamber $-\mathcal{V}$ in terms of the restricted simple coroots $(S'_{O_{\mathfrak{g}}})^\vee = \{\lambda_1^\vee, \dots, \lambda_m^\vee\}$:

- If $\mathfrak{g} = A_r$ $(r \geq 4)$, then $R'_{O_{\mathfrak{g}}} = BC_2$ and

$$\gamma_1 = \lambda_1^\vee + \lambda_2^\vee,$$
$$\gamma_2 = \lambda_1^\vee + 2\lambda_2^\vee$$

  where $\lambda_1^\vee(= \lambda_1^V)$ and $\lambda_2^\vee(= \lambda_2^V/2)$ are basis elements of $(R'_{O_{\mathfrak{g}}})^\vee$ (see Remark 2.3.12).

- If $\mathfrak{g} = A_3$, then $R'_{O_{\mathfrak{g}}} = C_2$ and

$$\gamma_1 = \lambda_1^\vee + \lambda_2^\vee,$$
$$\gamma_2 = \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee.$$

- If $\mathfrak{g} = A_2$ or C_r $(r \geq 2)$, then $R'_{O_{\mathfrak{g}}} = BC_1$, and $\gamma = \gamma_1 = \lambda^\vee$ where $\lambda^\vee(= \lambda^V/2)$ means the basis element of $(R'_{O_{\mathfrak{g}}})^\vee$ (hence $\langle \lambda, \lambda^\vee \rangle = 1$; see Remark 2.3.12).

- If $\mathfrak{g} = B_{r\geq 4}$ or $D_{r\geq 5}$, then $R'_{O_{\mathfrak{g}}} = B_4$ and

$$\gamma_1 := \lambda_1^\vee + \lambda_2^\vee + \lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_2 := \lambda_1^\vee + 2\lambda_2^\vee + 2\lambda_3^\vee + \lambda_4^\vee,$$
$$\gamma_3 := \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + \frac{3}{2}\lambda_4^\vee,$$
$$\gamma_4 := \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + 2\lambda_4^\vee.$$

- If $\mathfrak{g} = B_3$, then $R'_{O_{\mathfrak{g}}} = B_3$ and

$$\gamma_1 := \lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee,$$
$$\gamma_2 := \lambda_1^\vee + 2\lambda_2^\vee + \lambda_3^\vee,$$
$$\gamma_3 := \lambda_1^\vee + 2\lambda_2^\vee + \frac{3}{2}\lambda_3^\vee.$$

- If $\mathfrak{g} = D_4$, then $R'_{O_{\mathfrak{g}}} = D_4$ and

$$\gamma_1 := \lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_2 := \lambda_1^\vee + 2\lambda_2^\vee + \lambda_3^\vee + \lambda_4^\vee,$$
$$\gamma_3 := \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee + \lambda_3^\vee + \frac{1}{2}\lambda_4^\vee,$$
$$\gamma_4 := \frac{1}{2}\lambda_1^\vee + \lambda_2^\vee + \frac{1}{2}\lambda_3^\vee + \lambda_4^\vee.$$

- If $\mathfrak{g}$ is of an exceptional type other than G_2, then $R'_{O_\mathfrak{g}} = F_4$ and

$$\begin{aligned}\gamma_1 &:= 2\lambda_1^\vee + 3\lambda_2^\vee + 4\lambda_3^\vee + 2\lambda_4^\vee,\\ \gamma_2 &:= 3\lambda_1^\vee + 6\lambda_2^\vee + 8\lambda_3^\vee + 4\lambda_4^\vee,\\ \gamma_3 &:= 2\lambda_1^\vee + 4\lambda_2^\vee + 6\lambda_3^\vee + 3\lambda_4^\vee,\\ \gamma_4 &:= \lambda_1^\vee + 2\lambda_2^\vee + 3\lambda_3^\vee + 2\lambda_4^\vee.\end{aligned}$$

- If $\mathfrak{g} = G_2$, then $R'_{O_\mathfrak{g}} = G_2$ and

$$\gamma_1 := 2\lambda_1^\vee + 3\lambda_2^\vee, \quad \gamma_2 := \lambda_1^\vee + 2\lambda_2^\vee.$$

4.1 Colored Cones of Simple Embeddings

Consider a G-variety X with an open spherical G-orbit and let $Y \subset X$ be a closed G-orbit which is projective. If X admits a G-linearized ample line bundle, then by Proposition 2.3.7, for the normalization $\pi : X^{nor} \to X$, $\pi^{-1}(Y)$ is a closed G-orbit in X^{nor}. Moreover, since Y is simply connected and the restriction $\pi : \pi^{-1}(Y) \to Y$ is a G-equivariant finite morphism, we have $\pi^{-1}(Y) \simeq Y$.

Now assume that the open orbit of X is isomorphic to $O_\mathfrak{g}$, and put

$$X_Y := \{x \in X^{nor} : \overline{G \cdot x} \supset \pi^{-1}(Y)\}.$$

Then X_Y is open in X^{nor}, and in fact X_Y is a simple $O_\mathfrak{g}$-embedding with a unique closed orbit $\pi^{-1}(Y) \simeq Y$. In the notation of Section 3.2, for a B'-color $\mathcal{D} \in \mathcal{D}(O_\mathfrak{g})$, since $\mathrm{Stab}_G(\mathcal{D})$ is a parabolic subgroup containing B', we can write

$$\mathrm{Stab}_G(\mathcal{D}) = P'_{I'(\mathcal{D})}, \quad I'(\mathcal{D}) \subset S'.$$

If $I(\mathcal{D}) := I'(\mathcal{D}) \circ Ad_g$ and w_0 is a representative of the longest element in W_G with respect to B, then since $P'_{I'(\mathcal{D})} = g \cdot P_{I(\mathcal{D})} \cdot g^{-1}$, Lemma 2.3.6 implies that

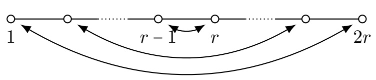
$$\text{(the isotropy group of } Y \text{ containing } B) = \bigcap_{\mathcal{D} \in \mathcal{D}(O_\mathfrak{g}) \smallsetminus \mathcal{F}(X_Y)} (w_0 \cdot P_{I(\mathcal{D})} \cdot w_0^{-1})^-. \tag{4.1}$$

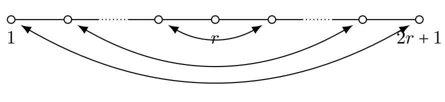
Remark 4.1.1. 1. $I'(\mathcal{D})$ is explicitly given in Remark 2.3.12 when $\mathfrak{g} = A_r$, $r \geq 2$. In other cases, by Theorem 2.3.11 and the discussion in Section 3.2, the color map $\epsilon : \mathcal{D}(O_\mathfrak{g}) \to \frac{1}{2}(S'_\mathfrak{g})^\vee$ is bijective and

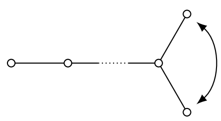
$$I'(\mathcal{D}_i) = \{\alpha'_j \in S' : \overline{\alpha'_j} = \lambda_i\}$$

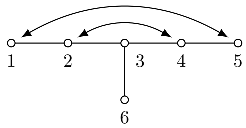
where $\mathcal{D}_i$ is the color $\epsilon^{-1}(\lambda_i^\vee/2)$.

2. The action of w_0 on the set of roots is well-known. See [6, PLATE I–IX]. Indeed, under Ad_{w_0}, $\alpha_j \in S$ is sent to $-\tau(\alpha_j)$ where $\tau : S \to S$ is a map given by the following diagram involutions:

   - If $\mathfrak{g} = A_{2r}$ $(r \geq 1)$: 

   - If $\mathfrak{g} = A_{2r+1}$ $(r \geq 1)$: 

   - If $\mathfrak{g} = D_{2r+1}$ $(r \geq 2)$: 

- If $\mathfrak{g} = E_6$: 

- Otherwise, $\tau = id$.

Since our compactifications are equipped with G-equivariant finite morphisms

$$\begin{aligned}
\mathbf{C}_{\mathfrak{g}} &\hookrightarrow \mathrm{Chow}_{1,2}(\mathbb{P}(\mathfrak{g}), \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)) \\
&\to \{\text{hypersurfaces of degree } (2,2) \text{ in } \mathbb{P}(\mathfrak{g}^*) \times \mathbb{P}(\mathfrak{g}^*)\} \\
&\hookrightarrow \mathbb{P}\left((\mathrm{Sym}^2 \mathfrak{g})^{\otimes 2}\right), \\
\mathbf{H}_{\mathfrak{g}} &\to \mathrm{Hilb}_{2m+1}(\mathbb{P}(\mathfrak{g}), \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)) \\
&\hookrightarrow \mathrm{Gr}\left(\mathrm{Sym}^N(\mathfrak{g}^*), M\right) \\
&\subset \mathbb{P}\left(\bigwedge^M \mathrm{Sym}^N(\mathfrak{g}^*)\right)
\end{aligned}$$

for some positive integers M and N (see [20, Chapter I]), $\mathbf{C}_{\mathfrak{g}}$ and $\mathbf{H}_{\mathfrak{g}}$ are equipped with G-linearized ample line bundles. Since $\mathbf{CoC}_{\mathfrak{g}} \subset \mathbf{CoC}(\mathbb{P}(\mathfrak{g}))$, $\mathbf{CoC}_{\mathfrak{g}}$ also admits a G-linearized ample line bundle. Thus the previous discussion can be applied, and in particular, by Proposition 2.3.7, we may identify orbits in $\mathbf{C}_{\mathfrak{g}}$ (respectively, in $\mathbf{H}_{\mathfrak{g}}$ and in $\mathbf{CoC}_{\mathfrak{g}}$) with orbits in its normalization.

Now the colored cone of X^{nor} can be easily computed in the following cases:

1. The case where the color map $\epsilon : \mathcal{D}(O_{\mathfrak{g}}) \to \frac{1}{2}(S'_{\mathfrak{g}})^\vee$ is injective and X is simple. This assumption is satisfied in the following cases:

   (a) The case where $\mathfrak{g}$ is not of type A or C, and $X = \mathbf{C}_{\mathfrak{g}}$. Indeed, ϵ is injective by Theorem 2.3.11 and Theorem 3.2.1, and $\mathbf{C}_{\mathfrak{g}}^{nor}$ is simple by Corollary 3.5.5. For example, if $\mathfrak{g} = B_r$ with $r \geq 3$, then we have

$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_1,\alpha_3}, \quad I_i = \{\alpha_i\}, \quad Ad_{w_0} = -id$$

   by Theorem 2.2.7, Table 3.1 and [6, Plate II]. It means that $(w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = P_{\alpha_i}$ for each i, hence

$$\mathcal{F}(\mathbf{C}^{nor}(Z_{B_r})) = \begin{cases} \{\mathcal{D}_2, \mathcal{D}_4\} & (\text{if } r \geq 4), \\ \{\mathcal{D}_2\} & (\text{if } r = 3) \end{cases}$$

   After similar computations using the following list of the isotropy groups and $(w_0 \cdot P_{I_i} \cdot w_0^{-1})^-$, we obtain Table 3.2.

   - $\mathfrak{g} = D_r$ with $r \geq 6$:

$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_1,\alpha_3}, \quad (w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = P_{\alpha_i}, \quad \forall i = 1, 2, 3, 4.$$

   - $\mathfrak{g} = D_5$:

$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_1,\alpha_3}, \quad (w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = \begin{cases} P_{\alpha_i}, & \text{if } i = 1, 2, 3; \\ P_{\alpha_4,\alpha_5}, & \text{if } i = 4. \end{cases}$$

   - $\mathfrak{g} = D_4$:

$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_1,\alpha_3,\alpha_4}, \quad (w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = P_{\alpha_i}, \quad \forall i = 1, 2, 3, 4.$$

- $\mathfrak{g} = E_6$:
$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_3},$$
$$(w_0 \cdot P_{I_1} \cdot w_0^{-1})^- = P_{\alpha_1, \alpha_5}, \quad (w_0 \cdot P_{I_2} \cdot w_0^{-1})^- = P_{\alpha_2, \alpha_4},$$
$$(w_0 \cdot P_{I_3} \cdot w_0^{-1})^- = P_{\alpha_3}, \quad (w_0 \cdot P_{I_4} \cdot w_0^{-1})^- = P_{\alpha_6}.$$
- $\mathfrak{g} = E_7$:
$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_5},$$
$$(w_0 \cdot P_{I_1} \cdot w_0^{-1})^- = P_{\alpha_2}, \quad (w_0 \cdot P_{I_2} \cdot w_0^{-1})^- = P_{\alpha_4},$$
$$(w_0 \cdot P_{I_3} \cdot w_0^{-1})^- = P_{\alpha_5}, \quad (w_0 \cdot P_{I_4} \cdot w_0^{-1})^- = P_{\alpha_6}.$$
- $\mathfrak{g} = E_8$:
$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_2},$$
$$(w_0 \cdot P_{I_1} \cdot w_0^{-1})^- = P_{\alpha_7}, \quad (w_0 \cdot P_{I_2} \cdot w_0^{-1})^- = P_{\alpha_3},$$
$$(w_0 \cdot P_{I_3} \cdot w_0^{-1})^- = P_{\alpha_2}, \quad (w_0 \cdot P_{I_4} \cdot w_0^{-1})^- = P_{\alpha_1}.$$
- $\mathfrak{g} = F_4$:
$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_3},$$
$$(w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = P_{\alpha_i}, \quad \forall i = 1, \dots, 4.$$
- $\mathfrak{g} = G_2$:
$$\mathrm{Stab}_G(\mathcal{L}_B) = P_{\alpha_1},$$
$$(w_0 \cdot P_{I_i} \cdot w_0^{-1})^- = P_{\alpha_i}, \quad \forall i = 1, 2.$$

(b) The case where $\mathfrak{g}$ is of exceptional type and X is either $\mathbf{H}_{\mathfrak{g}}$ and $\mathbf{CoC}_{\mathfrak{g}}$. In this case, ϵ is injective by Theorem 3.2.1 and Theorem 2.3.11, and X is simple by Corollary 3.5.5. As before, using Table 3.6 and the above list of $(w_0 \cdot P_{I_i} \cdot w_0^{-1})^-$, we obtain Tables 3.3–3.4 for exceptional Lie algebras.

2. The case where $\mathfrak{g} = A_r$ for $r \geq 4$ and $X = \mathbf{C}_{A_r}$. Recall that in this case, $\mathbf{C}_{A_r}^{nor} \simeq \mathbf{H}_{A_r}^{nor} \simeq \mathbf{CoC}_{A_r}^{nor}$ by the discussion of Subsection 3.1.2. Furthermore, $\mathbf{C}_{A_r}^{nor}$ is a simple O_{A_r}-embedding with a unique closed orbit $G/P_{\alpha_1, \alpha_2, \alpha_{r-1}, \alpha_r}$ by Lemma 3.5.1. By the equation 4.1 and Remark 2.3.12, we have
$$\mathcal{F}(\mathbf{C}_{A_r}^{nor}) = \varnothing,$$
hence
$$\mathcal{C}(\mathbf{C}_{A_r}^{nor}) = \mathcal{V} = \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2 \rangle$$
since $\mathbf{C}_{A_r}^{nor}$ is projective.

Together with Remark 3.3.6, Theorem 3.2.2 follows, except when $\mathfrak{g}$ is A_3 or of type B or D. These are covered in the rest of this chapter.

4.2 The Case of Orthogonal Lie Algebras

In this section, we compute the colored data of $\mathbf{H}_{\mathfrak{g}}^{nor}$ and $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ for $\mathfrak{g}$ of type B or D, using the colored cone of $\mathbf{C}_{\mathfrak{g}}^{nor}$. First, let us prove the following lemma.

Lemma 4.2.1. *Assume that $\mathfrak{g}$ is not of type A or C. In $\mathbf{C}_{\mathfrak{g}}^{nor}$, each G-orbit represented by planar reducible conics corresponds to a colored face of codimension 1 in the colored cone of $\mathbf{C}_{\mathfrak{g}}^{nor}$.*

Proof. Let $\mathcal{O} \subset \mathbf{C}_{\mathfrak{g}}$ be an orbit represented by planar reducible conics, and $\mathcal{P}$ a plane contained in $Z_{\mathfrak{g}}$ such that reducible conics in $\mathcal{P}$ represent points in $\mathcal{O}$. Recall that $\mathcal{O}$ is not a closed orbit, since $\mathfrak{g}$ is not of type A or C (Corollary 3.5.5). Since non-planarity and smoothness are open conditions, every boundary point of $\mathcal{O}$ should be represented by a planar reducible conic or a double line. If a planar reducible conic C represents a boundary point of $\mathcal{O}$, then the plane spanned by C is in the closure of the G-orbit containing $\mathcal{P}$ in the space of planes in $Z_{\mathfrak{g}}$. However since the space of planes in $Z_{\mathfrak{g}}$ is the disjoint union of rational homogeneous spaces by [23, Theorem 4.9], the plane spanned by C is indeed G-conjugate to $\mathcal{P}$, which is a contradiction to Corollary 3.5.4. Therefore the boundary of $\mathcal{O}$ consists of double lines, and the same statement holds for $\pi^{-1}(\mathcal{O})$, which is a G-orbit by Proposition 2.3.7, where $\pi : \mathbf{C}_{\mathfrak{g}}^{nor} \to \mathbf{C}_{\mathfrak{g}}$ is the normalization map. Since double lines form the unique closed orbit in $\mathbf{C}_{\mathfrak{g}}$, the desired statement follows. □

As shown in Corollary 3.5.5 and Table 3.6, $\mathbf{H}_{\mathfrak{g}}^{nor}$ and $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ contain at least two closed G-orbits when $\mathfrak{g}$ is of type B or D. For a closed G-orbit Y of $\mathbf{H}_{\mathfrak{g}}^{nor}$ (of $\mathbf{CoC}_{\mathfrak{g}}^{nor}$, respectively), define

$$\mathbf{H}_Y := \{x \in \mathbf{H}_{\mathfrak{g}}^{nor} : Y \subset \overline{G\cdot x}\}, \quad (\mathbf{CoC}_Y := \{x \in \mathbf{CoC}_{\mathfrak{g}}^{nor} : Y \subset \overline{G\cdot x}\}, \quad \text{respectively}).$$

As remarked in Section 2.3, it is a simple $O_{\mathfrak{g}}$ embedding with its unique closed orbit Y, and the colored cones of $\mathbf{H}_Y$ (of $\mathbf{CoC}_Y$, respectively) for all closed orbits Y are exactly the maximal elements of the colored fan of $\mathbf{H}_{\mathfrak{g}}^{nor}$ (of $\mathbf{CoC}_{\mathfrak{g}}^{nor}$, respectively).

4.2.1 High Rank Cases

Now assume that $\mathfrak{g} = B_r$ $(r \geq 4)$ or D_r $(r \geq 5)$ so that the restrictive root system is $R'_{O_{\mathfrak{g}}} = B_4$ (Theorem 3.2.1). By Corollary 3.5.5 and Table 3.6, $\mathbf{H}_{\mathfrak{g}}^{nor}$ has exactly two closed orbits and they are represented by planar double lines. Thus the colored fan of $\mathbf{H}_{\mathfrak{g}}^{nor}$ consists of two maximal colored cones and their colored faces by Lemma 2.3.4. For $i = 1, 2$, let Y_i be the closed orbit in $\mathbf{H}_{\mathfrak{g}}^{nor}$ represented by double lines on the ith plane $\mathcal{P}_i$ in Table 3.5 (in the row $\mathfrak{g}$). By Table 3.6 and the equation (4.1) in Section 4.1, we have

$$\mathcal{F}(\mathbf{H}_{Y_1}) = \{\mathcal{D}_2, \mathcal{D}_4\}, \quad \mathcal{F}(\mathbf{H}_{Y_2}) = \{\mathcal{D}_2\}.$$

Let $\mathcal{O}_i \subset \mathbf{C}_{\mathfrak{g}}^{nor}$ be the G-orbit containing planar reducible conics in the ith plane in Table 3.5 for $i = 1, 2$. By Lemma 4.2.1, each $\mathcal{O}_i$ corresponds to a colored face of dimension 3, and such a face contains extremal rays generated by $-\gamma_2$ and $-\gamma_4$ by the list of colored faces in Section 3.2. Note that if a 1-dimensional colored face $\mathbb{Q}_{\geq 0} \cdot (-\gamma)$ of $\mathbf{C}_{\mathfrak{g}}^{nor}$ is contained in the colored face corresponding to $\mathcal{O}_i$ for some i, then the G-stable divisor corresponding to $\mathbb{Q}_{\geq 0} \cdot (-\gamma)$ contains $\mathcal{O}_i$ in $\mathbf{C}_{\mathfrak{g}}^{nor}$. Since $\mathcal{O}_i$ is contained in the open subset where the morphism $FC^{nor} : \mathbf{H}_{\mathfrak{g}}^{nor} \to \mathbf{C}_{\mathfrak{g}}^{nor}$ is an isomorphism, and Y_i is contained in the closure of its preimage under FC^{nor}, the strict transform of the divisor corresponding to $\mathbb{Q}_{\geq 0} \cdot (-\gamma)$

contains Y_i. In other words, the colored cone of $\mathbf{H}_{Y_i}$ contains $\mathbb{Q}_{\geq 0} \cdot (-\gamma)$ as an extremal ray. Therefore the colored cone of $\mathbf{H}_{Y_1}$ contains

$$\mathcal{C}_1 := \mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, \lambda_2^\vee, \lambda_4^\vee \rangle \quad (\ni -\gamma_3 = -\gamma_4 + \lambda_4^\vee/2)$$

and the colored cone of $\mathbf{H}_{Y_2}$ contains $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, \lambda_2^\vee \rangle$. If $-\gamma_1$ is contained in the colored cone of $\mathbf{H}_{Y_1}$, then it contains the valuation cone $\mathcal{V}$, which is a contradiction since $\mathbf{H}_{Y_1}$ is not complete. Hence the colored cone of $\mathbf{H}_{Y_2}$ contains

$$\mathcal{C}_2 := \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, \lambda_2^\vee \rangle.$$

By the definition of a colored fan, especially 1(a) and 4(b) in Definition 2.3.2, $(\mathcal{C}_1, \{\mathcal{D}_2, \mathcal{D}_4\})$ and $(\mathcal{C}_2, \{\mathcal{D}_2\})$ are colored cones of $\mathbf{H}_{Y_1}$ and $\mathbf{H}_{Y_2}$, respectively.

Similarly, by Corollary 3.5.5, $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ contains two closed orbits, say $\tilde{Y}_i$, such that $CH^{nor}(\tilde{Y}_i) = Y_i$ for $i = 1, 2$. As before, the equation 4.1 and Table 3.6 imply that

$$\mathcal{F}(\mathbf{CoC}_{\tilde{Y}_1}) = \{\mathcal{D}_4\}, \quad \mathcal{F}(\mathbf{CoC}_{\tilde{Y}_2}) = \varnothing.$$

Furthermore, since we have a morphism $CH^{nor} : \mathbf{CoC}_{\mathfrak{g}}^{nor} \to \mathbf{H}_{\mathfrak{g}}^{nor}$, we have

$$\mathcal{C}(\mathbf{CoC}_{\tilde{Y}_i}) \subset \mathcal{C}_i, \quad i = 1, 2.$$

Since $\mathbf{CoC}_{\mathfrak{g}}^{nor}$ is complete, and since

$$\mathcal{C}_1 \cap \mathcal{V} = \mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_3, -\gamma_4, -\gamma_1 - \gamma_3 \rangle, \quad \mathcal{C}_2 \cap \mathcal{V} = \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3 \rangle,$$

we see that

$$\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3, \lambda_4^\vee \rangle \subset \mathcal{C}(\mathbf{CoC}_{\tilde{Y}_1}), \quad \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3 \rangle \subset \mathcal{C}(\mathbf{CoC}_{\tilde{Y}_2}).$$

Since the union of the cones $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3, \lambda_4^\vee \rangle$ and $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, -\gamma_1 - \gamma_3 \rangle$ contains the valuation cone $\mathcal{V} = \mathbb{Q}_{\geq 0}\langle -\gamma_1, \ldots, -\gamma_4 \rangle$, we conclude that the equalities hold.

4.2.2 The Case of D_4

Next, we compute the colored fans when $\mathfrak{g} = D_4$, i.e. when $R'_{O_{\mathfrak{g}}} = D_4$. By Corollary 3.5.5 and Table 3.6, $\mathbf{H}_{D_4}^{nor}$ has three closed G-orbits, and each of them consists of planar double lines. For each $i \in \{1, 3, 4\}$ and the colored face

$$\mathcal{C}'_i := \mathbb{Q}_{\geq 0}\langle -\gamma_j : j \in \{1, 2, 3, 4\} \smallsetminus \{i\} \rangle$$

of the colored cone of $\mathbf{C}_{D_4}^{nor}$, there is a B-stable plane $\mathcal{P}_i$ in Z_{D_4} such that planar reducible conics in $\mathcal{P}_i$ belong to the G-orbit in $\mathbf{C}_{D_4}^{nor}$ corresponding to $\mathcal{C}'_i$ by Corollary 3.5.4 and Lemma 4.2.1.

For each i, let Y_i be the closed orbit in $\mathbf{H}_{D_4}^{nor}$ containing double lines in $\mathcal{P}_i$. Since all of Y_i have same isotropy group $P_{\alpha_1, \alpha_3, \alpha_4}$, the equation (4.1) in Section 4.1 shows that

$$\mathcal{F}(\mathbf{H}_{Y_1}) = \mathcal{F}(\mathbf{H}_{Y_3}) = \mathcal{F}(\mathbf{H}_{Y_4}) = \{\mathcal{D}_2\}.$$

As in the previous section, since the strict transform (via FC^{nor}) of the divisor corresponding to each extremal ray of $\mathcal{C}'_i$ contains Y_i, the colored cone of $\mathbf{H}_{Y_i}$ contains

$$\mathcal{C}_i := \mathbb{Q}_{\geq 0}\langle \mathcal{C}'_i, \lambda_2^\vee \rangle = \mathbb{Q}_{\geq 0}\langle -\gamma_j, \lambda_2^\vee : j \in \{1, 2, 3, 4\} \smallsetminus \{i\} \rangle.$$

A straightforward computation using the relation $-\gamma_1+2\gamma_2-\gamma_3-\gamma_4 = \lambda_2^\vee$ shows that for $\sum_{i=1}^4 a_i\cdot(-\gamma_i)\in\mathcal{V}$, $\alpha_i\in\mathbb{Q}_{\geq 0}$, we have

$$\sum_{i=1}^4 a_i\cdot(-\gamma_i)\in\begin{cases}\mathcal{C}_1 & \text{if and only if } a_1\leq a_3 \text{ and } a_1\leq a_4;\\ \mathcal{C}_3 & \text{if and only if } a_3\leq a_1 \text{ and } a_3\leq a_4;\\ \mathcal{C}_4 & \text{if and only if } a_4\leq a_1 \text{ and } a_4\leq a_3.\end{cases}$$

By 1(a) and 4(b) in Definition 2.3.2, the colored cone of $\mathbf{H}_{Y_i}$ is $(\mathcal{C}_i, \{\mathcal{D}_2\})$ for each i.

Similarly, by Corollary 3.5.5, $\mathbf{CoC}_{D_4}^{nor}$ contains three closed orbits $\tilde{Y}_i$ such that $CH^{nor}(\tilde{Y}_i) = Y_i$ for each $i = 1, 2, 3$. Since the isotropy group of $\tilde{Y}_i$ is B by Table 3.6, by the equation 4.1, we have

$$\mathcal{F}(\mathbf{CoC}_{\tilde{Y}_i}) = \varnothing, \quad \forall i = 1, 2, 3.$$

Furthermore, since $CH^{nor}: \mathbf{CoC}_{D_4}^{nor}\to\mathbf{H}_{D_4}^{nor}$ sends $\mathbf{CoC}_{\tilde{Y}_i}$ to $\mathbf{H}_{Y_i}$, we have

$$\mathcal{C}(\mathbf{CoC}_{\tilde{Y}_i})\subset\mathcal{C}_i.$$

Since $\mathbf{CoC}_{D_4}^{nor}$ is projective, we conclude that

$$\mathcal{C}(\mathbf{CoC}_{\tilde{Y}_i}) = \mathcal{C}_i\cap\mathcal{V} = \begin{cases}\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_3, -\gamma_4, -\gamma_1-\gamma_3-\gamma_4\rangle & (i=1)\\ \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, -\gamma_1-\gamma_3-\gamma_4\rangle & (i=2)\\ \mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_3, -\gamma_1-\gamma_3-\gamma_4\rangle & (i=3).\end{cases}$$

4.2.3 The Case of B_3

Finally, consider the case where $\mathfrak{g} = B_3$. By Corollary 3.5.5, $\mathbf{H}_{B_3}^{nor}$ contains two closed orbits, say Y_1 and Y_3. By the equation (4.1) in Section 4.1, the colors of $\mathbf{H}_{Y_1}$ and $\mathbf{H}_{Y_3}$ are given by

$$\mathcal{F}(\mathbf{H}_{Y_1}) = \mathcal{F}(\mathbf{H}_{Y_3}) = \{\mathcal{D}_2\},$$

since P_{α_1,α_3} is an isotropy group of each of Y_1 and Y_3 (Table 3.6).

For each $i = 1, 3$, let $\mathcal{O}_i$ be the G-orbit in $\mathbf{C}_{B_3}^{nor}$ corresponding to the colored face $\mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_2\rangle$. If one of the closed orbits, say Y_j, is contained in the closures of the preimages of both $\mathcal{O}_1$ and $\mathcal{O}_3$ under FC^{nor}, then the colored cone of $\mathbf{H}_{Y_j}$ contains $\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_3, \lambda_2^\vee\rangle$, which is a contradiction since $\mathbf{H}_{Y_j}$ is not complete. Therefore we may assume that for each $i = 1, 3$, Y_i is contained in the closure of the preimage of $\mathcal{O}_i$. In other words, the colored cone of $\mathbf{H}_{Y_i}$ contains

$$\mathcal{C}_i := \mathbb{Q}_{\geq 0}\langle -\gamma_i, -\gamma_2, \lambda_2^\vee\rangle.$$

By 1(a) and 4(b) in Definition 2.3.2, $(\mathcal{C}_1, \{\mathcal{D}_2\})$ and $(\mathcal{C}_2, \{\mathcal{D}_2\})$ are maximal colored cones in the colored fan of $\mathbf{H}_{B_3}^{nor}$.

By Corollary 3.5.5, $\mathbf{CoC}_{B_3}^{nor}$ contains two closed orbits $\tilde{Y}_i$, and we have $CH^{nor}(\tilde{Y}_i) = Y_i$ for $i = 1, 3$. By Table 3.6, the isotropy groups of $\tilde{Y}_i$ is B, hence by the equation 4.1,

$$\mathcal{F}(\mathbf{CoC}_{\tilde{Y}_1}) = \mathcal{F}(\mathbf{CoC}_{\tilde{Y}_3}) = \varnothing.$$

Since $\mathbf{CoC}_{B_3}^{nor}$ is projective, and since $CH^{nor}: \mathbf{CoC}_{B_3}^{nor}\to\mathbf{H}_{B_3}^{nor}$ sends $\mathbf{CoC}_{\tilde{Y}_i}$ to $\mathbf{H}_{Y_i}$, we have

$$\mathcal{C}(\mathbf{CoC}_{\tilde{Y}_i}) = \mathcal{C}_i\cap\mathcal{V} = \begin{cases}\mathbb{Q}_{\geq 0}\langle -\gamma_1, -\gamma_2, -\gamma_1-\gamma_3\rangle & (i=1)\\ \mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_3, -\gamma_1-\gamma_3\rangle & (i=3).\end{cases}$$

4.3 The Case of Special Linear Lie Algebras

In this section, we compute the colored fan in the case $\mathfrak{g} = A_3$ to complete the proof of Theorem 3.2.2.

Suppose that $\mathfrak{g} = A_r$, $r \geq 2$. We use the notation of Subsection 3.1.2. Namely, $\mathfrak{g} = \mathfrak{sl}(V)$ for a vector space V of dimension $r+1$, and $Z_{\mathfrak{g}}$ is equipped with two projections $p := p_1 : Z_{\mathfrak{g}} \to \mathbb{P}(V)$ and $q := p_r : Z_{\mathfrak{g}} \to \mathbb{P}(V^*)$. Recall that $\mathbf{H}_{\mathfrak{g}}^{nor}(\simeq \mathbf{C}_{\mathfrak{g}}^{nor} \simeq \mathbf{CoC}_{\mathfrak{g}}^{nor})$ parametrizes (1, 1)-conics. That is, for a conic C representing a point in $\mathbf{H}_{\mathfrak{g}}$, C is a reduced scheme, and $p(C)$ and $q(C)$ are lines on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. Thus we have a G-equivariant morphism

$$p \times q : \mathbf{H}_{\mathfrak{g}} \to \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*), \quad [C] \mapsto ([p(C)], [q(C)]).$$

Observe that $\mathrm{Gr}(2, V^*)$ can be identified with $\mathrm{Gr}(r-1, V)$. Thus under the diagonal $G = SL(V)$-action, the orbit structure of $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*) \simeq \mathrm{Gr}(2, V) \times \mathrm{Gr}(r-1, V)$ is given as follows:

- If $r = 2$ (i.e. $\dim V = 3$), then $\mathrm{Gr}(2, V) \times \mathrm{Gr}(1, V)$ consists of the following two orbits:
  - An open orbit
  $$\{([W_2], [W_1]) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(1, V) : W_2 \cap W_1 = 0\}.$$
  - A unique closed orbit of codimension 1
  $$\{([W_2], [W_1]) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(1, V) : W_2 \supset W_1\}(\simeq G/B).$$
- If $r = 3$ (i.e. $\dim V = 4$), then $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V)$ consists of the following three orbits:
  - An open orbit
  $$\{([W_2], [W_2']) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V) : W_2 \cap W_2' = 0\}.$$
  - A codimension 1 orbit
  $$\{([W_2], [W_2']) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V) : \dim(W_2 \cap W_2') = 1\}.$$
  - A unique closed orbit of codimension 4
  $$\mathrm{diag}(\mathrm{Gr}(2, V)) := \{([W_2], [W_2']) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V) : W_2 = W_2'\}(\simeq G/P_{\alpha_2}).$$
- If $r \geq 4$ (i.e. $\dim V \geq 5$), then $\mathrm{Gr}(2, V) \times \mathrm{Gr}(r-1, V)$ consists of the following three orbits:
  - An open orbit
  $$\{([W_2], [W_{r-1}]) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(r-1, V) : W_2 \cap W_{r-1} = 0\}.$$
  - A codimension 1 orbit
  $$\{([W_2], [W_{r-1}]) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(r-1, V) : \dim(W_2 \cap W_{r-1}) = 1\}.$$
  - A unique closed orbit of codimension 4
  $$\mathrm{Fl}_{2, r-1}(V) := \{([W_2], [W_{r-1}]) \in \mathrm{Gr}(2, V) \times \mathrm{Gr}(r-1, V) : W_2 \subset W_{r-1}\}(\simeq G/P_{\alpha_2, \alpha_{r-1}}).$$

In each case, $\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V^*)$ contains an open G-orbit O and a unique closed G-orbit. We claim that $p\times q$ is bijective over O. That is, for 2-dimensional subspaces $W_2\subset V$ and $U_2\subset V^*$, if $([W_2],[U_2])\in O$, then there is a unique $[C]\in\mathbf{H}_{\mathfrak{g}}$ such that $p(C)=\mathbb{P}W_2$ and $q(C)=\mathbb{P}U_2$. Indeed, since $([W_2],[U_2])\in O$, for every $w\in W_2$, we have $\langle w,U_2\rangle\not\equiv 0$ where $\langle\,,\,\rangle : V\times V^*\to\mathbb{C}$ is the natural pairing. If $\{u_1,u_2\}$ is a basis of U_2, then since each u_i defines a hyperplane in V, there exist $w_1,w_2\in W_2$ such that $\langle w_1,u_2\rangle=\langle w_2,u_1\rangle=0$. Since $\langle w_i,u_i\rangle\neq 0$, we may assume that $\langle w_i,u_j\rangle=\delta_{ij}$. Now in $Z_{\mathfrak{g}}\simeq \mathrm{Fl}_{1,r}(V)$, we have

$$\begin{aligned}
p^{-1}(\mathbb{P}W_2)\cap q^{-1}(\mathbb{P}U_2) &= \{[(a_1w_1+a_2w_2)\otimes(b_1u_1+b_2u_2)] : a_i,b_i\in\mathbb{C},\ \langle a_1w_1+a_2w_2, b_1u_1+b_2u_2\rangle=0\}\\
&= \{[(a_1w_1+a_2w_2)\otimes(b_1u_1+b_2u_2)] : a_i,b_i\in\mathbb{C},\ a_1b_1+a_2b_2=0\}\\
&\simeq (a_1b_1+a_2b_2=0) \text{ in } \mathbb{P}^1_{a_1,a_2}\times\mathbb{P}^1_{b_1,b_2},
\end{aligned}$$

which is a smooth conic. Hence the claim follows, and in particular, $p\times q$ induces a birational morphism $\mathbf{H}^{nor}_{\mathfrak{g}}\to\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V^*)$.

Now suppose that $\mathfrak{g}=A_3$ so that $\dim V=4$ and we have a G-equivariant birational morphism

$$p\times q : \mathbf{H}^{nor}_{A_3}\to\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V^*)\simeq\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V).$$

Recall that $\mathbf{H}^{nor}_{A_3}$ and $\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V)$ have unique closed orbits, isomorphic to G/B (Lemma 3.5.1) and $\mathrm{Gr}(2,V)\simeq G/P_{\alpha_2}$, respectively. In particular, $p\times q$ is not an isomorphism.

On the other hand, by the equation (4.1) and Remark 2.3.12, the set of colors of $\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V)$ is one of

$$\{\mathcal{D}_1\},\quad \{\mathcal{D}_1,\mathcal{D}_2^+\},\quad \{\mathcal{D}_1,\mathcal{D}_2^-\}.$$

However, since $\mathcal{V}$, $\lambda_1^\vee$ and $\lambda_2^\vee$ spans a cone which is not strictly convex, the colored cone of $\mathrm{Gr}(2,V)\times\mathrm{Gr}(2,V)$ must be

$$(\mathbb{Q}_{\geq 0}\langle -\gamma_1,\lambda_1^\vee\rangle,\{\mathcal{D}_1\}).$$

Similarly, the set of colors of $\mathbf{H}^{nor}_{A_3}$ is one of

$$\varnothing,\quad \{\mathcal{D}_2^+\},\quad \{\mathcal{D}_2^-\}.$$

By the existence of the morphism $p\times q$, we have $\mathcal{F}(\mathbf{H}^{nor}_{A_3})\subset\{\mathcal{D}_1\}$, which means $\mathcal{F}(\mathbf{H}^{nor}_{A_3})=\varnothing$. Therefore the colored cone of $\mathbf{H}^{nor}_{A_3}$ is $(\mathcal{V},\varnothing)$.

Chapter 5. Applications

In this chapter, we present several applications of our main Theorems 3.2.1 and 3.2.2.

5.1 Classical Descriptions

In Subsection 3.1.1, we see that the space of conics on $Z_{\mathfrak{g}}$ for $\mathfrak{g}$ of type C is isomorphic to the Grassmannian. In this section, we describe the space of conics in terms of projective geometry, especially when $\mathfrak{g}$ is of type A or G_2.

5.1.1 Type A: Blow-up of the Product of Grassmannians

Suppose that $\mathfrak{g} = A_r = \mathfrak{sl}(V)$, $r \geq 2$. In Section 4.3, we have seen that there is a G-equivariant birational morphism

$$p \times q : \mathbf{H}_{\mathfrak{g}}^{nor} \to \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*), \quad [C] \mapsto ([p(C)], [q(C)]),$$

and $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*)$ is a simple $O_{\mathfrak{g}}$-embedding with unique closed orbit $\mathrm{Fl}_{2, r-1}(V) \simeq G/P_{\alpha_2, \alpha_{r-1}}$. Moreover, if $r = 3$, then its colored cone is $(\mathbb{Q}_{\geq 0}\langle -\gamma_1, \lambda_1^\vee \rangle, \{\mathcal{D}_1\})$ by the discussion in Section 4.3. Similarly, by the equation 4.1 (Section 4.1) and Remark 2.3.12, one can show that the set of colors of $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*)$ is $\varnothing$ if $r = 2$ and $\{\mathcal{D}_1\}$ if $r \geq 4$. Therefore its colored cone is

$$(\mathbb{Q}_{\geq 0}\langle -\gamma \rangle, \varnothing) \quad \text{if } r = 2, \text{ and } \quad (\mathbb{Q}_{\geq 0}\langle -\gamma_1, \lambda_1^\vee \rangle, \{\mathcal{D}_1\}) \quad \text{if } r \geq 3.$$

In particular, if $r = 2$, then the morphism $\mathbf{H}_{\mathfrak{g}}^{nor} \to \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*)$ is an isomorphism. If $r \geq 3$, then the colored cone $(\mathcal{V}, \varnothing)$ defines a unique complete $O_{\mathfrak{g}}$-embedding admitting a G-equivariant birational morphism to $\mathrm{Gr}(2, V) \times \mathrm{Gr}(2, V^*)$, which is not an isomorphism. By Theorem 3.2.2, we obtain the following proposition.

Proposition 5.1.1. *Suppose that $\mathfrak{g} = \mathfrak{sl}(V)$ for a vector space V of dimension $r + 1 \geq 3$.*

1. *If $r = 2$, then $\mathbf{H}_{\mathfrak{g}}^{nor}$ is isomorphic to $Gr(2, V) \times Gr(2, V^*)$.*
2. *If $r \geq 3$, then $\mathbf{H}_{\mathfrak{g}}^{nor}$ is isomorphic to the blow-up of $Gr(2, V) \times Gr(2, V^*)$ along the partial flag variety $Fl_{2, r-1}(V)$.*

Remark 5.1.2. If $\mathfrak{g}$ is of type A, then $\mathbf{H}_{\mathfrak{g}}^{nor}$ is smooth by Proposition 5.1.1, and its colored cone is $(\mathcal{V}, \varnothing)$ by Theorem 3.2.2. It means that $\mathbf{H}_{\mathfrak{g}}^{nor}$ is a wonderful variety in the sense of Remark 2.2.10.

5.1.2 Type G_2: Cayley Grassmannian

By Theorem 3.2.2 for $\mathfrak{g} = G_2$, the morphism $FC^{nor} : FC^{nor} : \mathbf{H}_{G_2}^{nor} \to \mathbf{C}_{G_2}^{nor}$ is an isomorphism. In fact, since $O_{\mathfrak{g}} = G/G^\sigma$ (Theorem 3.2.1), we see that the colored fan of $\mathbf{H}_{G_2}^{nor}$ is same with that of [37, Theorem 4.1.(xi)], which represents an 8-dimensional smooth symmetric variety of Picard number 1, called the *Cayley Grassmannian CG*. Here, for the complexifixed Octonion algebra $\mathbb{O}_{\mathbb{C}}$ and its imaginary part Im $\mathbb{O}_{\mathbb{C}}$, the Cayley Grassmannian CG is defined to be the subset of $\mathrm{Gr}(3, \mathrm{Im}\ \mathbb{O}_{\mathbb{C}})$ consisting of the imaginary parts of the 4-dimensional subalgebras of $\mathbb{O}_{\mathbb{C}}$. Geometry of CG is investigated in [28], and moreover, it has been observed that CG parametrizes conics on Z_{G_2} in [29, p. 1784]. Furthermore, since the open orbit O_{G_2} in CG admits a wonderful compactification, and since a colored cone of a wonderful variety is of form $(\mathcal{V}, \varnothing)$, we see that $\mathbf{CoC}_{G_2}$ is the wonderful compactification of O_{G_2} by Theorem 3.2.2.

5.2 Conjugacy Classes of Conics

In this section, we describe the G-conjugacy classes of conics in adjoint varieties. Since the G-conjugacy classes of conics correspond to G-orbits in $\mathbf{H}_{\mathfrak{g}}$, we study G-orbits in $\mathbf{H}_{\mathfrak{g}}$. First, we count their number, using Lemma 2.3.4, Proposition 2.3.7 and Theorem 3.2.2.

Corollary 5.2.1. *The number of G-orbits in $\mathbf{C}_{\mathfrak{g}}$ is*

$$\begin{cases} 4 & \text{if } \mathfrak{g} = A_r \text{ with } r \geq 3; \\ 2 & \text{if } \mathfrak{g} = A_2,\ C_r \text{ with } r \geq 2; \\ 11 & \text{if } \mathfrak{g} = B_r \text{ with } r \geq 4,\ D_r \text{ with } r \geq 5; \\ 7 & \text{if } \mathfrak{g} = B_3, E_6, E_7, E_8, F_4; \\ 15 & \text{if } \mathfrak{g} = D_4; \\ 3 & \text{if } \mathfrak{g} = G_2. \end{cases}$$

The number of G-orbits in $\mathbf{H}_{\mathfrak{g}}$ is

$$\begin{cases} 4 & \text{if } \mathfrak{g} = A_r \text{ with } r \geq 3; \\ 2 & \text{if } \mathfrak{g} = A_2,\ C_r \text{ with } r \geq 2; \\ 15 & \text{if } \mathfrak{g} = B_r \text{ with } r \geq 4,\ D_r \text{ with } r \geq 5; \\ 9 & \text{if } \mathfrak{g} = B_3, E_6, E_7, E_8, F_4; \\ 21 & \text{if } \mathfrak{g} = D_4; \\ 3 & \text{if } \mathfrak{g} = G_2. \end{cases}$$

To figure out which conjugacy class of conics is associated to each colored face, we need to analyze geometry of singular conics in more detail. The following proposition shows that reducible conics form a prime divisor in the spaces of conics.

Proposition 5.2.2. *Assume that $\mathfrak{g}$ is not of type C. Then each of $\mathbf{H}_{\mathfrak{g}}$ and $\mathbf{C}_{\mathfrak{g}}$ contains a prime divisor parametrizing all reducible conics and whose general points are represented by non-planar reducible conics.*

Proof. It is enough to consider the Hilbert scheme. In the notation of Subsection 2.2.2, define

$$\mathcal{K}_o := \begin{cases} \mathcal{C}_o^{\alpha_1} \times \mathcal{C}_o^{\alpha_r} & \text{if } \mathfrak{g} = A_r,\ r \geq 2; \\ (\mathcal{C}_o^{\alpha_{j_0}} \times \mathcal{C}_o^{\alpha_{j_0}}) \smallsetminus \operatorname{diag}(\mathcal{C}_o^{\alpha_{j_0}}) & \text{otherwise.} \end{cases}$$

Since $\mathfrak{g}$ is not of type C, each element of $\mathcal{K}_o$ represents a reducible conic, via a morphism

$$u : \mathcal{K}_o \to \mathbf{H}_{\mathfrak{g}}, \quad ([T_o\mathcal{L}_1], [T_o\mathcal{L}_2]) \mapsto [\mathcal{L}_1 \cup \mathcal{L}_2]$$

where $\mathcal{L}_i$'s are lines passing through o. Moreover, u is finite onto its image (bijective if $\mathfrak{g} = A_r$, and 2-to-1 otherwise), and its image is the locus of reducible conics singular at o. Thus $\dim u(\mathcal{K}_o) = \dim \mathcal{K}_o = 2 \cdot (n-1)$. Since for each $g \in G$, the translation $g \cdot u(\mathcal{K}_o)$ is the locus of reducible conics singular at $g \cdot o$, the locus of reducible conics is given by $G \cdot \mathcal{K}_o$, which is irreducible and whose dimension is equal to

$$\dim \mathcal{K}_o + \dim Z_{\mathfrak{g}} = (2n-2) + (2n+1) = 4n - 1.$$

In particular, it is a prime divisor.

To see that general reducible conics are non-planar, we may assume that $\mathfrak{g}$ is not of type A (as no planar conic is parametrized by $\mathbf{H}_{A_r}$; see Subsection 3.1.2). Suppose that $([T_o\mathcal{L}_1], [T_o\mathcal{L}_2]) \in \mathcal{K}_o$ and put $[C] := u([T_o\mathcal{L}_1], [T_o\mathcal{L}_2])$, i.e. $C = \mathcal{L}_1 \cup \mathcal{L}_2$. Then C is planar if and only if the secant line joining $[T_o\mathcal{L}_1]$ and $[T_o\mathcal{L}_2]$ is contained in $\mathcal{C}_o^{\alpha_{j_0}} \subset \mathbb{P}(T_oZ_{\mathfrak{g}})$ (cf. [23, Section 4.3]). Therefore if every reducible conic is planar in $Z_{\mathfrak{g}}$, then the secant variety of $\mathcal{C}_o^{\alpha_{j_0}}$ coincides with $\mathcal{C}_o^{\alpha_{j_0}}$ itself, which is a contradiction (see [16, Section 6.3]). □

Next, we count the conjugacy classes of reducible conics when $\mathfrak{g}$ is not of type C. If $\mathfrak{g} = A_r$, $r \geq 2$, then these are easy to count. In fact, if $\mathfrak{g} = A_2$, then by Theorem 3.2.2 and Proposition 5.2.2, $\mathbf{H}_{A_2} \setminus O_{A_2}$ is a single orbit represented by reducible conics. If $\mathfrak{g} = A_r$, $r \geq 3$, then by Theorem 3.2.2 and Proposition 5.2.2, the locus of reducible conics consists of two orbits.

Thus we focus on the cases where $\mathfrak{g}$ is not of type A or C. To do this, define P^{ss} to be the semi-simple part of the isotropy group P at $o \in Z_{\mathfrak{g}}$. Then the Dynkin diagram of P^{ss} can be obtained by removing α_{j_0} in the Dynkin diagram of G, and P^{ss} acts transitively on the space of lines passing through o (Theorem 2.2.7). With respect to this action, let $Q \subset P^{ss}$ be an isotropy group. We may choose Q as the parabolic subgroup of P^{ss} generated by the complement of $N(\alpha_{j_0})$ by [23, Theorem 4.8]. Also choose a line l_0 passing through o such that $\mathrm{Stab}_{P^{ss}}(l_0) = Q$.

Lemma 5.2.3. *Assume that $\mathfrak{g}$ is not of type A or C. Then we have the following:*

1. *The number of G-conjugacy classes of reducible conics in $Z_{\mathfrak{g}}$ is equal to*

$$|W_{P^{ss},Q} \backslash W_{P^{ss}} / W_{P^{ss},Q}| - 1,$$

   i.e. the number of double cosets $W_{P^{ss},Q} \cdot w \cdot W_{P^{ss},Q}$ ($w \in W_{P^{ss}}$) minus 1. For each $\mathfrak{g}$, the number of double cosets is given in Table 5.1.

2. *The G-stable prime divisor of $\mathbf{C}_{\mathfrak{g}}^{nor}$ given in Proposition 5.2.2 corresponds to the following colored face:*

$$\begin{cases} \mathbb{Q}_{\geq 0} \cdot (-\gamma_2) & \text{if } \mathfrak{g} \text{ is } B_r\ (r \geq 3),\ D_r\ (r \geq 4) \text{ or } G_2; \\ \mathbb{Q}_{\geq 0} \cdot (-\gamma_4) & \text{if } \mathfrak{g} \text{ is } E_r\ (r = 6, 7, 8) \text{ or } F_4. \end{cases}$$

Proof. First of all, the number of double cosets can be easily computed from the diagram of the parabolic subgroup Q in P^{ss}, for instance by using the description of Weyl groups ([6, Plate I-IX]) and recipies for the Hasse diagrams ([1, Chapter 4]).

We claim that the number of double cosets minus 1 is an upper bound of the number of conjugacy classes of reducible conics. Note that each G-conjugacy class of reducible conics in $Z_{\mathfrak{g}}$ has a representative which is singular at o. Moreover, if two reducible conics singular at o are G-conjugate, then they must be P-conjugate since o is their unique singular point. In other words, there is a bijection between

$$\{G\text{-conjugacy classes of reducible conics in } Z_{\mathfrak{g}}\}$$

and

$$\{P\text{-conjugacy classes of reducible conics in } Z_{\mathfrak{g}} \text{ singular at } o\}.$$

Each P-conjugacy class of reducible conics singular at o has a representative of form $l_0 \cup l$ for some line l such that $l_0 \cap l = \{o\}$. Since $\mathrm{Stab}_{P^{ss}}(l_0) = Q$ and P^{ss} acts on the space of lines containing o transitively,

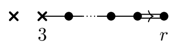
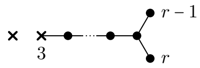
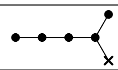
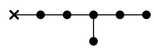
$\mathfrak{g}$	Diagram of $Q \subset P^{ss}$	$\lvert W_{P^{ss},Q} \backslash W_{P^{ss}} / W_{P^{ss},Q} \rvert$
B_r $(r \geq 4)$		6
B_3		4
D_r $(r \geq 6)$		6
D_5		6
D_4		8
E_6		4
E_7		4
E_8		4
F_4		4
G_2		2

Table 5.1: Number of double cosets of $W_{P^{ss},Q}$ in $W_{P^{ss}}$ for the parabolic $Q \subset P^{ss}$.

the number of P-conjugacy classes of such $l_0 \cup l$ is at most the number of Q-orbits in $P^{ss}/Q - \{e \cdot Q\}$, which is equal to $\lvert W_{P^{ss},Q} \backslash W_{P^{ss}} / W_{P^{ss},Q} \rvert - 1$ by the generalized Bruhat decomposition ([6, Chapter IV.2.5]).

Now let us show the equalities. Let $D \subset \mathbf{C}_{\mathfrak{g}}^{nor}$ be the inverse image of the G-stable prime divisor given in Proposition 5.2.2. Then D corresponds to a 1-dimensional colored face of the colored cone of $\mathbf{C}_{\mathfrak{g}}^{nor}$. By its definition, the ray corresponding to D is contained in the colored faces corresponding to orbits defined by reducible conics. Moreover, if $\mathfrak{g} \neq B_3, G_2$, then the number of conjugacy classes of planes contained in $Z_{\mathfrak{g}}$ (Table 3.5) and the number of colored faces of codimension 1 are same. Thus by Lemma 4.2.1, if $\mathfrak{g} \neq B_3, G_2$, then the ray corresponding to D is contained in the intersection of all colored faces of codimension 1 in the colored cone of $\mathbf{C}_{\mathfrak{g}}^{nor}$.

If $\mathfrak{g} = G_2$, then $\mathbf{C}_{\mathfrak{g}}^{nor}$ has only one colored extremal ray, and the number of colored faces containing it is 2. Thus the statement follows from Lemma 2.3.4.

If $\mathfrak{g}$ is exceptional but $\neq G_2$, then $\mathbf{C}_{\mathfrak{g}}^{nor}$ has two colored extremal ray $\mathbb{Q}_{\geq 0} \cdot (-\gamma_1)$ and $\mathbb{Q}_{\geq 0} \cdot (-\gamma_4)$, and the numbers of colored faces containing them are 5 and 4, respectively. By Lemma 2.3.4, the number of colored faces containing the ray determined by D is at most 4. Therefore D corresponds to the ray $\mathbb{Q}_{\geq 0} \cdot (-\gamma_4)$ and the statement follows.

If $\mathfrak{g} = D_4$, the intersection of all colored faces of codimension 1 is $\mathbb{Q}_{\geq 0} \cdot (-\gamma_2)$. Thus D corresponds to $\mathbb{Q}_{\geq 0} \cdot (-\gamma_2)$ and the number of colored faces containing it is 8.

If $\mathfrak{g}$ is B_r $(r \geq 4)$ or D_r $(r \geq 5)$, then the intersection of colored faces of codimension 1 is $\mathbb{Q}_{\geq 0}\langle -\gamma_2, -\gamma_4 \rangle$. Thus D corresponds to either $\mathbb{Q}_{\geq 0}\langle -\gamma_2 \rangle$ or $\mathbb{Q}_{\geq 0}\langle -\gamma_4 \rangle$, and each of them is contained in 6 and 7 number of colored faces, respectively. By Table 5.1, D corresponds to $\mathbb{Q}_{\geq 0}\langle -\gamma_2 \rangle$ and the upper bound is attained.

From now on, assume that $\mathfrak{g} = B_3$, and recall the list of the colored faces given in Section 3.2. Then the space of planes in Z_{B_3} is homogeneous, and the unique B-stable plane in Z_{B_3}, say $\mathcal{P}$, is given in Table 3.5. By Corollary 3.5.4, planar contact conics and planar reducible conic form single orbits $\mathcal{O}_{PC}$ and $\mathcal{O}_{PR}$ in $\mathbf{C}_{B_3}$, respectively. Note that the stabilizer of a given planar contact conic in $\mathcal{P}$ is contained in $\mathrm{Stab}_G(\mathcal{P})$. Since the space of smooth conics in $\mathcal{P} \simeq \mathbb{P}^2$ is 5-dimensional, and since the map

$\mathrm{Stab}_G(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$ is surjective by Corollary 3.5.4, an isotropy group of $\mathcal{O}_{PC}$ in G is of codimension 5 in $\mathrm{Stab}_G(\mathcal{P})$. Since $\mathrm{Stab}_G(\mathcal{P}) = P_{\alpha_3}$, this implies that

$$\dim \mathcal{O}_{PC} = \dim G/P_{\alpha_3} + 5 = 11 = \dim O_{B_3} - 1.$$

That is, $\overline{\mathcal{O}_{PC}}$ is a G-stable divisor in $\mathbf{C}_{B_3}$. Hence its inverse image D_{PC} in $\mathbf{C}_{B_3}^{nor}$ corresponds to a 1-dimensional colored face of the colored cone. Observe that the colored cone of $\mathbf{C}_{B_3}^{nor}$ has three 1-dimensional colored faces. Thus there is a 1-dimensional colored face not corresponding to D nor D_{PC} and it corresponds to a G-stable divisor defined by non-planar contact conics.

Suppose that the number of conjugacy classes of reducible conics in Z_{B_3} is strictly less than 3, which is the number of double cosets minus 1. Then D corresponds to a ray $\mathbb{Q}_{\geq 0} \cdot (-\gamma_i)$ for some $i \in \{1, 3\}$, and the number of conjugacy classes of reducible conics is equal to 2. It means that the orbit $\mathcal{O}_{NPR}$ corresponding to D is indeed a unique orbit of non-planar reducible conics, hence the number of conjugacy classes of non-planar contact conics is

$$7 - 1(\text{twistor}) - 1(\text{non-planar reducible}) - 3(\text{planar contact/reducible or double line}) = 2$$

by Corollary 5.2.1. Thus one of the 2-dimensional colored faces corresponds to a conjugacy class $\mathcal{O}$ of non-planar contact conics. However, since any 2-dimensional colored faces contain one of rays corresponding to D or D_{PC}, $\mathcal{O}$ is contained in the boundary of $\mathcal{O}_{NPR}$ or the boundary of $\mathcal{O}_{PC}$. This is a contradiction since both smoothness and non-planarity are open conditions. Therefore the number of conjugacy classes of reducible conics is 3, and D corresponds to $\mathbb{Q}_{\geq 0} \cdot (-\gamma_2)$. □

Theorem 5.2.4. *1. Let $\mathfrak{g} = C_r$, $r \geq 2$. Then $\mathbf{C}_{C_r} \simeq \mathbf{H}_{C_r}$ consists of two orbits: one for twistor conics, and one for non-planar contact conics.*

2. Let $\mathfrak{g} = A_2$. Then both $\mathbf{C}_{A_2}$ and $\mathbf{H}_{A_2}$ consist of two orbits: one for twistor conics, and one for non-planar reducible conics.

3. Let $\mathfrak{g} = A_r$, $r \geq 3$. Then both $\mathbf{C}_{A_r}$ and $\mathbf{H}_{A_r}$ consist of four orbits: one for twistor conics, one for non-planar contact conics, and two for non-planar reducible conics.

4. Let $\mathfrak{g}$ be B_r with $r \geq 4$ or D_r with $r \geq 5$.

(a) $\mathbf{C}_{\mathfrak{g}}$ consists of eleven orbits: one for twistor conics, two for non-planar contact conics, two for planar contact conics, three for non-planar reducible conics, two for planar reducible conics, and one for double lines.

(b) $\mathbf{H}_{\mathfrak{g}}$ consists of fifteen orbits: one for twistor conics, two for non-planar contact conics, two for planar contact conics, three for non-planar reducible conics, two for planar reducible conics, three for non-planar double lines, and two for planar double lines.

5. Let $\mathfrak{g}$ be either B_3 or of an exceptional type other than G_2.

(a) $\mathbf{C}_{\mathfrak{g}}$ consists of seven orbits: one for twistor conics, one for non-planar contact conics, one for planar contact conics, two for non-planar reducible conics, one for planar reducible conics, and one for double lines.

(b) $\mathbf{H}_{\mathfrak{g}}$ consists of nine orbits: one for twistor conics, one for non-planar contact conics, one for planar contact conics, two for non-planar reducible conics, one for planar reducible conics, two for non-planar double lines, and one for planar double lines.

6. *Let* $\mathfrak{g} = D_4$*.*

   (a) $\mathbf{C}_{D_4}$ *consists of fifteen orbits: one for twistor conics, three for non-planar contact conics, three for planar contact conics, four for non-planar reducible conics, three for planar reducible conics, and one for double lines.*

   (b) $\mathbf{H}_{D_4}$ *consists of twenty one orbits: one for twistor conics, three for non-planar contact conics, three for planar contact conics, four for non-planar reducible conics, three for planar reducible conics, four for non-planar double lines, and three for planar double lines.*

7. *If* $\mathfrak{g} = G_2$*, then both* $\mathbf{C}_{G_2}$ *and* $\mathbf{H}_{G_2}$ *consist of three orbits: one for twistor conics, one for non-planar reducible conics, and one for non-planar double lines. In particular, every smooth conic in* Z_{G_2} *is a twistor conic.*

Proof. For the case where $\mathfrak{g}$ is of type A or C, see Subsection 3.1.1 (type C) and the discussion after Proposition 5.2.2. Then the numbers follow immediately from Corollary 5.2.1.

Thus we may assume that $\mathfrak{g}$ is not of type A or C. The statements for the Chow schemes follow from Lemma 3.3.2, Corollary 3.5.4, Corollary 3.5.5, Lemma 5.2.3 and Corollary 5.2.1.

For $\mathbf{H}_\mathfrak{g}$, recall that the morphism $FC : \mathbf{H}_\mathfrak{g} \to \mathbf{C}_\mathfrak{g}$ gives a bijective correspondence for orbits of smooth conics and reducible conics (Remark 3.1.3). Therefore

$$\#(\text{orbits in } \mathbf{H}_\mathfrak{g}) - \#(\text{orbits in } \mathbf{C}_\mathfrak{g}) = \#(\text{orbits of double lines in } \mathbf{H}_\mathfrak{g}) - 1$$

(as $\mathbf{C}_\mathfrak{g}$ contains a unique orbit of double lines). The left hand side can be computed by Corollary 5.2.1, and then the statements follow from Corollary 3.5.4. □

In Figures 5.1–5.8, we visualize the orbit structure of $\mathbf{H}_\mathfrak{g}$ for each $\mathfrak{g}$. In the following graphs, each vertex represents a G-orbit in $\mathbf{H}_\mathfrak{g}$ (or a conjugacy class of conics in $Z_\mathfrak{g}$), and each edge $\begin{smallmatrix} A \\ | \\ B \end{smallmatrix}$ means that $B \subset \overline{A}$ in $\mathbf{H}_\mathfrak{g}$, and there is no other orbit B' such that $B \subset \overline{B'} \subset \overline{A}$. We also use the abbreviations (N)PC, (N)PR, and (N)PD for (non-)planar contact, (non-)planar reducible, and (non-)planar double, respectively.

In particular, the orbit structures of $\mathbf{H}_\mathfrak{g}$ for B_3 and for exceptional Lie algebras $\neq G_2$ are different, although the numbers of conjugacy classes are same (Theorem 5.2.4).

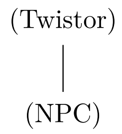


Figure 5.1: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = C_r$ $(r \geq 2)$.

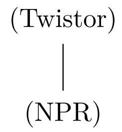


Figure 5.2: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = A_2$.

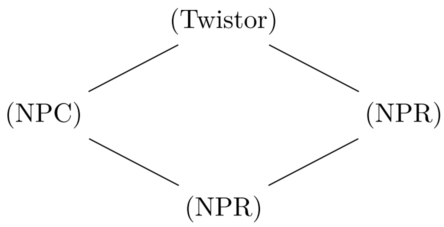


Figure 5.3: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = A_r$ $(r \geq 3)$.

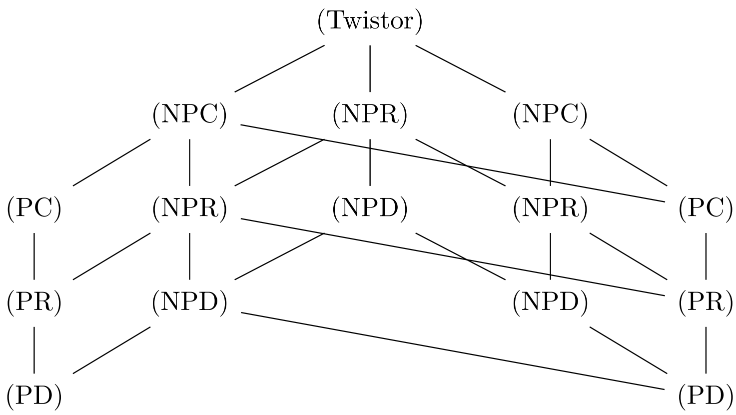


Figure 5.4: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = B_r$ $(r \geq 4)$ or D_r $(r \geq 5)$.

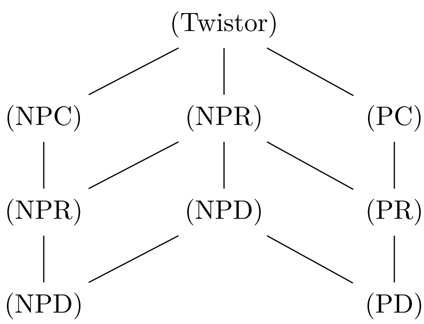


Figure 5.5: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = B_3$.

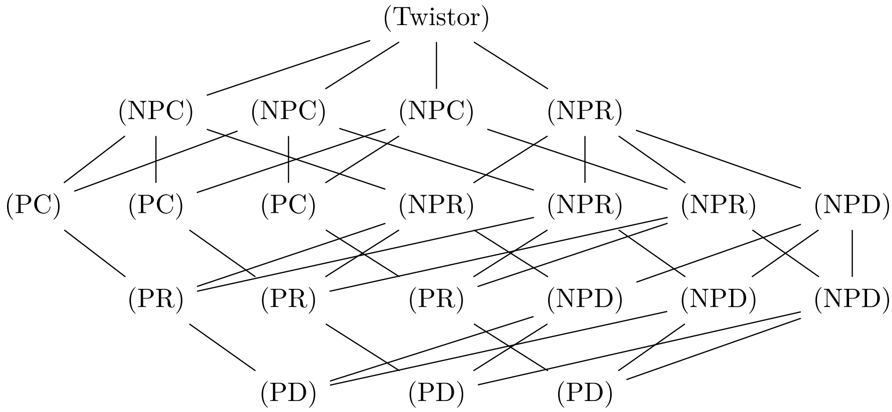


Figure 5.6: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = D_4$.

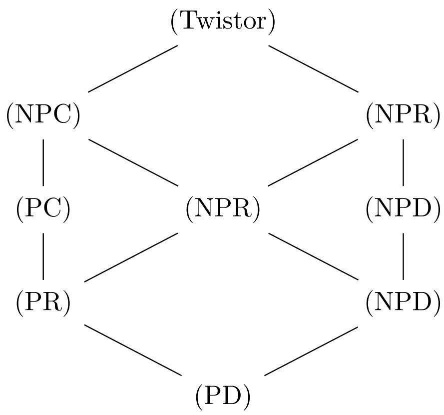


Figure 5.7: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = E_r$ $(r = 6, 7, 8)$ or F_4.

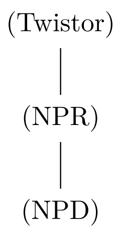


Figure 5.8: Orbit structure of $\mathbf{H}_{\mathfrak{g}}$ for $\mathfrak{g} = G_2$.

5.3 Smoothness of Hilbert Schemes

Let us prove smoothness of $\mathbf{H}_{\mathfrak{g}}^{nor}$ and determine the singular locus of $\mathbf{C}_{\mathfrak{g}}^{nor}$.

Corollary 5.3.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra.*

1. *$\mathbf{H}_{\mathfrak{g}}^{nor}$ is smooth. Moreover, the anticanonical line bundle of $\mathbf{H}_{\mathfrak{g}}^{nor}$ is*
   - *not globally generated if $\mathfrak{g} = B_r$ ($r \neq 5$ and $r \geq 3$), D_r ($r \neq 6$ and $r \geq 4$), E_8;*
   - *globally generated but not ample if $\mathfrak{g} = B_5$, D_6, E_7, F_4;*
   - *ample if $\mathfrak{g} = A_r$ ($r \geq 2$), C_r ($r \geq 2$), E_6, G_2.*
2. *$\mathbf{C}_{\mathfrak{g}}^{nor}$ is*
   - *not $\mathbb{Q}$-Gorenstein if $\mathfrak{g} = B_r$ ($r \neq 5$ and $r \geq 3$), D_r ($r \geq 4$), E_6, E_7, E_8;*
   - *Gorenstein Fano with terminal singularities but not $\mathbb{Q}$-factorial if $\mathfrak{g} = B_5$, F_4;*
   - *smooth Fano if $\mathfrak{g} = A_r$ ($r \geq 2$), C_r ($r \geq 2$), G_2.*

In particular, the singular locus of $\mathbf{C}_{\mathfrak{g}}^{nor}$ is equal to the subset formed by double lines if $\mathfrak{g} \neq G_2$.

In fact, the smoothness of $\mathbf{H}_{\mathfrak{g}}^{nor}$ follows from [10, Proposition 3.6]. In the following paragraphs, we present another proof using spherical geometry.

Proof. First, if $\mathfrak{g} = C_r$ ($r \geq 2$), then $\mathbf{H}_{\mathfrak{g}} (\simeq \mathbf{C}_{\mathfrak{g}})$ is the Grassmannian (Subsection 3.1.1), hence the statement follows. If $\mathfrak{g} = G_2$, then $\mathbf{H}_{\mathfrak{g}}^{nor} (\simeq \mathbf{C}_{\mathfrak{g}}^{nor})$ is the Cayley Grassmannian, which is a smooth Fano variety (Subsection 5.1.2). In the case where $\mathfrak{g} = A_r$ ($r \geq 2$), $\mathbf{H}_{\mathfrak{g}}^{nor} (\simeq \mathbf{C}_{\mathfrak{g}}^{nor})$ is the blow-up of the product of two Grassmannians (Subsection 5.1.1). In particular, it is smooth. Observe that its colored fan is $(\mathcal{V}, \varnothing)$ (Theorem 3.2.2), and so $\mathbf{H}_{\mathfrak{g}}^{nor}$ is Fano by [35, Theorem 2.1].

Next, consider $\mathbf{H}_{\mathfrak{g}}^{nor}$ for other $\mathfrak{g}$. We apply Ruzzi's smoothness criterion for symmetric varieties in [37, Theorem 3.2]. Observe that the notation of [37] is slightly different from ours, since we use a different definition for the restricted root system. In our setting (Section 2.3, Section 3.2), the criterion can be formulated as follows.

Theorem 5.3.2 ([37, Theorem 3.2])**.** *Assume that $\mathfrak{g}$ is not of type A or C (hence ϵ is injective and $R'_{O_{\mathfrak{g}}}$ is a reduced root system). Let X be a simple $O_{\mathfrak{g}}$-embedding, and assume that its unique closed orbit Y is projective. For the standard Levi factor L of $\bigcap_{\mathcal{D} \in \mathcal{D}(O_{\mathfrak{g}}) \setminus \mathcal{F}(X)} Stab_G(\mathcal{D})$, let $R_{L,\sigma}$ be the sub-root system of $R'_{O_{\mathfrak{g}}}$ spanned by the roots of L. The simple factors of $R_{L,\sigma}$ are denoted by $R^j_{L,\sigma}$ so that $R_{L,\sigma} = \prod_{j=1}^{p} R^j_{L,\sigma}$ for some integer p, and their simple roots $\{\lambda_i^j\}_i := S'_{O_{\mathfrak{g}}} \cap R^j_{L,\sigma}$ are indexed as in [32].*

Then X is smooth if and only if the following conditions are satisfied:

1. *For every j, $R^j_{L,\sigma}$ is of type A. Moreover, $\sum_{j=1}^{p}(l_j+1)$ is at most the rank of $R'_{O_{\mathfrak{g}}}$ where l_j is the rank of $R^j_{L,\sigma}$;*
2. *The cone $\mathcal{C}(X)$ is spanned by a basis $\mathcal{B}$ of $\frac{1}{2} \cdot \mathbb{Z}\langle (R'_{O_{\mathfrak{g}}})^{\vee} \rangle$, i.e. the half of the coroot lattice of the root system $R'_{O_{\mathfrak{g}}}$;*
3. *In the doubled weight lattice $2 \cdot (\mathbb{Z}\langle (R'_{O_{\mathfrak{g}}})^{\vee} \rangle)^*$, we can index the dual basis of $\mathcal{B}$ as*

$$\{y_1^1, \ldots, y_{l_1+1}^1, \ldots, y_1^q \ldots, y_{l_q+1}^q\}$$

for some integers $q (\geq p)$ and l_j so that *(a)* $\langle y^j_i, (2\lambda^h_k)^\vee\rangle$ *is* $= 1$ *if* $j = h$ *and* $i = k$*, and* $= 0$ *otherwise;*

(b) $y^j_i - \frac{i}{l_j+1} y^j_{l_{j+1}}$ *is the* i*th fundamental weight of* $R^j_{L,\sigma}$ *times* 2 *for* $1 \le j \le p$ *and* $1 \le i \le l_j$*.*

We claim that the simple spherical varieties defined by maximal colored cones in Table 3.3 satisfy the conditions in Theorem 5.3.2. Indeed, in each case, if π_i denotes the ith fundamental weight of $R'_{O_\mathfrak{g}}$, then $R_{L,\sigma}$, $\mathcal{B}$ and its dual basis $\{y^j_i\}$ can be chosen as follows.

1. $\mathfrak{g} = B_r$ $(r \ge 4)$ or D_r $(r \ge 5)$: For $(\mathbb{Q}_{\ge 0}\langle -\gamma_2, -\gamma_4, \lambda_2^\vee, \lambda_4^\vee\rangle, \{\mathcal{D}_2, \mathcal{D}_4\})$,

$$R_{L,\sigma} = \underset{\lambda_2}{\bullet}\ \underset{\lambda_4}{\bullet} = A_1 \times A_1,$$

$$\mathcal{B} = \left\{-\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \lambda_3^\vee - \frac{1}{2}\lambda_4^\vee, -\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{3}{2}\lambda_3^\vee - \lambda_4^\vee, \frac{1}{2}\lambda_2^\vee, \frac{1}{2}\lambda_4^\vee\right\},$$

$$y^1_1 := -4\pi_1 + 2\pi_2,\ y^1_2 := -6\pi_1 + 2\pi_3,\ y^2_1 := 2\pi_1 - 2\pi_3 + 2\pi_4,\ y^2_2 := 4\pi_1 - 2\pi_3.$$

   For $(\mathbb{Q}_{\ge 0}\langle -\gamma_1, -\gamma_2, -\gamma_4, \lambda_2^\vee\rangle, \{\mathcal{D}_2\})$,

$$R_{L,\sigma} = \underset{\lambda_2}{\bullet} = A_1,$$

$$\mathcal{B} = \left\{-\lambda_1^\vee - \lambda_2^\vee - \lambda_3^\vee - \frac{1}{2}\lambda_4^\vee, -\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \lambda_3^\vee - \frac{1}{2}\lambda_4^\vee, -\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{3}{2}\lambda_3^\vee - \lambda_4^\vee, \frac{1}{2}\lambda_2^\vee\right\},$$

$$y^1_1 := 2\pi_2 - 4\pi_3 + 4\pi_4,\ y^1_2 := 2\pi_1 - 6\pi_3 + 8\pi_4,\ y^2_1 := -2\pi_1 + 2\pi_3 - 2\pi_4,\ y^2_2 := 2\pi_3 - 4\pi_4.$$

2. $\mathfrak{g} = B_3$: For $(\mathbb{Q}_{\ge 0}\langle -\gamma_1, -\gamma_2, \lambda_2^\vee\rangle, \{\mathcal{D}_2\})$,

$$R_{L,\sigma} = \underset{\lambda_2}{\bullet} = A_1,$$

$$\mathcal{B} = \left\{-\lambda_1^\vee - \lambda_2^\vee - \frac{1}{2}\lambda_3^\vee, -\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{1}{2}\lambda_3^\vee, \frac{1}{2}\lambda_2^\vee\right\},$$

$$y^1_1 := 2\pi_2 - 4\pi_3,\ y^1_2 := 2\pi_1 - 4\pi_3,\ y^2_1 := -2\pi_1 + 2\pi_3.$$

   For $(\mathbb{Q}_{\ge 0}\langle -\gamma_2, -\gamma_3, \lambda_2^\vee\rangle, \{\mathcal{D}_2\})$,

$$R_{L,\sigma} = \underset{\lambda_2}{\bullet} = A_1,$$

$$\mathcal{B} = \left\{-\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{1}{2}\lambda_3^\vee, -\lambda_1^\vee - 2\lambda_2^\vee - \frac{3}{2}\lambda_3^\vee, \frac{1}{2}\lambda_2^\vee\right\},$$

$$y^1_1 := -4\pi_1 + 2\pi_2,\ y^1_2 := -6\pi_1 + 4\pi_3,\ y^2_1 := 2\pi_1 - 2\pi_3.$$

3. $\mathfrak{g} = D_4$: For each $i \in \{1, 3, 4\}$, let $j \ne k$ be distinct elements in $\{1, 3, 4\} \setminus \{i\}$. For $(\mathbb{Q}_{\ge 0}\langle -\gamma_2, -\gamma_j, -\gamma_k, \lambda_2^\vee\rangle, \{\mathcal{D}_2\})$,

$$R_{L,\sigma} = \underset{\lambda_2}{\bullet} = A_1,$$

$$\mathcal{B} = \left\{-\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{1}{2}\lambda_3^\vee - \frac{1}{2}\lambda_4^\vee, -\lambda_2^\vee - \frac{1}{2}\lambda_i^\vee - \frac{1}{2}\lambda_j^\vee - \lambda_k^\vee, -\lambda_2^\vee - \frac{1}{2}\lambda_i^\vee - \lambda_j^\vee - \frac{1}{2}\lambda_k^\vee, \frac{1}{2}\lambda_2^\vee\right\},$$

$$y^1_1 := 2\pi_2 - 4\pi_i,\ y^1_2 := 2\pi_1 + 2\pi_3 + 2\pi_4 - 8\pi_i,\ y^2_1 := 2\pi_i - 2\pi_j,\ y^2_2 := 2\pi_i - 2\pi_k.$$

4. $\mathfrak{g} = E_r$ $(r = 6, 7, 8)$ or F_4: For $(\mathbb{Q}_{\ge 0}\langle -\gamma_1, -\gamma_4, \lambda_1^\vee, \lambda_4^\vee\rangle, \{\mathcal{D}_1, \mathcal{D}_4\})$,

$$R_{L,\sigma} = \underset{\lambda_1}{\bullet}\ \underset{\lambda_4}{\bullet} = A_1 \times A_1,$$

$$\mathcal{B} = \left\{-\lambda_1^\vee - \frac{3}{2}\lambda_2^\vee - 2\lambda_3^\vee - \lambda_4^\vee, -\frac{1}{2}\lambda_1^\vee - \lambda_2^\vee - \frac{3}{2}\lambda_3^\vee - \lambda_4^\vee, \frac{1}{2}\lambda_1^\vee, \frac{1}{2}\lambda_4^\vee\right\},$$

$$y^1_1 := 2\pi_1 - 4\pi_2 + 2\pi_3,\ y^1_2 := -6\pi_2 + 4\pi_3,\ y^2_1 := 4\pi_2 - 4\pi_3 + 2\pi_4,\ y^2_2 := 8\pi_2 - 6\pi_3.$$

For the remaining statements, we use the well-known criteria for singularities of spherical varieties. Our main reference is [33]. Let us briefly explain the criteria together with the necessary data given in Table 5.2. Recall that the valuation cone $\mathcal{V}$ is a cone in the vector space $\mathbb{Q}\langle (R'_{O_\mathfrak{g}})^\vee\rangle$. By taking its negative dual $-\mathcal{V}^\vee := \{m \in \mathbb{Q}\langle R'_{O_\mathfrak{g}}\rangle : \langle m, \mathcal{V}\rangle \le 0\}$ and the embedding $\mathbb{Q}\langle R'_{O_\mathfrak{g}}\rangle = \chi(T'/T' \cap G^\sigma) \otimes \mathbb{Q} \hookrightarrow \chi(T') \otimes \mathbb{Q} = \mathbb{Q}\langle R'\rangle$, we obtain a cone in $\mathbb{Q}\langle R'\rangle$. We call the primitive elements of $-\mathcal{V}^\vee \cap \Lambda_{O_\mathfrak{g}} (= -\mathcal{V}^\vee \cap \chi(T'/T' \cap G^\sigma))$ in $\mathbb{Q}\langle R'\rangle$ the *spherical roots* of $O_\mathfrak{g}$. Now for each color $\mathcal{D} \in \mathcal{D}(O_\mathfrak{g})$, choose a simple root $\alpha' \in S'$ which is not a root of (the standard Levi part of) $\mathrm{Stab}_G(\mathcal{D})$. That is, α' 'moves' the divisor $\mathcal{D} \subset O_\mathfrak{g}$. Then we say that

$$\mathcal{D} \text{ is of } \textit{type} \begin{cases} \text{(a)} & \text{if } \alpha' \text{ is a spherical root;} \\ \text{(2a)} & \text{if } 2\alpha' \text{ is a spherical root;} \\ \text{(b)} & \text{otherwise,} \end{cases}$$

and the type of $\mathcal{D}$ does not depend on the choice of α'. If $\mathcal{D}$ is of type (a) or (2a), then we put $a_\mathcal{D} := 1$. If $\mathcal{D}$ is of type (b), then $\alpha_\mathcal{D}$ is defined as follows. Let $S'' \subset S'$ be the set of simple roots which are not roots of (the standard Levi part of) the stabilizer of the open B'-orbit in $O_\mathfrak{g}$, and $R'' \subset (R')^+$ the set of positive roots which are not generated by $S' \setminus S''$. Then for $\mathcal{D}$ of type (b) and α' moving $\mathcal{D}$, define

$$a_\mathcal{D} := \sum_{\beta' \in R''} \langle \beta' \,|\, \alpha' \rangle.$$

Now for an $O_\mathfrak{g}$-embedding X, its anti-canonical divisor can be written as a Weil divisor

$$-K_X = \sum_{G\text{-stable divisor } \mathcal{D} \subset X} \mathcal{D} + \sum_{\mathcal{D}_i \in \mathcal{D}(O_\mathfrak{g})} a_{\mathcal{D}_i} \cdot \overline{\mathcal{D}_i}.$$

See [33, Theorem 2.20] for details. The spherical roots, types and integers $a_\mathcal{D}$ of the colors of $O_\mathfrak{g}$ can be deduced from the Satake diagram (Table 3.1) and Theorem 2.3.11, and their list is given in Table 5.2.

Now the statement on the singularities of $\mathbf{C}_\mathfrak{g}^{nor}$ can be obtained from the criteria for $\mathbb{Q}$-factoriality ([33, Proposition 3.3]), ($\mathbb{Q}$-)Cartier divisors ([33, Proposition 4.2]) and terminal singularities ([33, Proposition 5.2]) Similarly, for the positivity of anti-canonical divisors of $\mathbf{H}_\mathfrak{g}^{nor}$ and $\mathbf{C}_\mathfrak{g}^{nor}$, the criteria for global generatedness and ampleness in [33, Proposition 2.19] can be applied. We omit the detailed computation. □

$\mathfrak{g}$	Spherical Roots of $O_{\mathfrak{g}}$	Type and Coefficient $a_{\mathcal{D}_i}$ of Color $\mathcal{D}_i$
$B_r\ (r \geq 5)$	$2\alpha'_i\ (1 \leq i \leq 3)$, $2\alpha'_4 + \cdots + 2\alpha'_r$	$\mathcal{D}_i\ (1 \leq i \leq 3)$: (2a), $a_{\mathcal{D}_i} = 1$ $\mathcal{D}_4$: (b), $a_{\mathcal{D}_4} = 2r - 7$
$B_r\ (r = 3, 4)$	$2\alpha'_i\ (1 \leq i \leq r)$	$\mathcal{D}_i\ (1 \leq i \leq r)$: (2a), $a_{\mathcal{D}_i} = 1$
$D_r\ (r \geq 6)$	$2\alpha'_i\ (1 \leq i \leq 3)$, $2\alpha'_4 + \cdots + 2\alpha'_{r-2} + \alpha'_{r-1} + \alpha'_r$	$\mathcal{D}_i\ (1 \leq i \leq 3)$: (2a), $a_{\mathcal{D}_i} = 1$ $\mathcal{D}_4$: (b), $a_{\mathcal{D}_4} = 2(r-4)$
D_5	$2\alpha'_i\ (1 \leq i \leq 3)$, $\alpha'_4 + \alpha'_5$	$\mathcal{D}_i\ (1 \leq i \leq 3)$: (2a), $a_{\mathcal{D}_i} = 1$ $\mathcal{D}_4$: (b), $a_{\mathcal{D}_4} = 2$
D_4	$2\alpha'_i\ (1 \leq i \leq 4)$	$\mathcal{D}_i\ (1 \leq i \leq 4)$: (2a), $a_{\mathcal{D}_i} = 1$
E_6	$\alpha'_1 + \alpha'_5$, $\alpha'_2 + \alpha'_4$, $2\alpha'_3$, $2\alpha'_6$,	$\mathcal{D}_1$: (b), $a_{\mathcal{D}_1} = 2$ $\mathcal{D}_2$: (b), $a_{\mathcal{D}_2} = 2$ $\mathcal{D}_3$: (2a), $a_{\mathcal{D}_3} = 1$ $\mathcal{D}_4$: (2a), $a_{\mathcal{D}_4} = 1$
E_7	$\alpha'_1 + 2\alpha'_2 + \alpha'_3$, $\alpha_3 + 2\alpha'_4 + \alpha'_7$, $2\alpha'_5$, $2\alpha'_6$	$\mathcal{D}_1$: (b), $a_{\mathcal{D}_1} = 4$ $\mathcal{D}_2$: (b), $a_{\mathcal{D}_2} = 4$ $\mathcal{D}_3$: (2a), $a_{\mathcal{D}_3} = 1$ $\mathcal{D}_4$: (2a), $a_{\mathcal{D}_4} = 1$
E_8	$2\alpha'_1$, $2\alpha'_2$, $2\alpha'_3 + 2\alpha'_4 + 2\alpha'_5 + \alpha'_6 + \alpha'_8$, $\alpha'_4 + 2\alpha'_5 + 2\alpha'_6 + 2\alpha'_7 + \alpha'_8$	$\mathcal{D}_1$: (b), $a_{\mathcal{D}_1} = 8$ $\mathcal{D}_2$: (b), $a_{\mathcal{D}_2} = 8$ $\mathcal{D}_3$: (2a), $a_{\mathcal{D}_3} = 1$ $\mathcal{D}_4$: (2a), $a_{\mathcal{D}_4} = 1$
F_4	$2\alpha'_i\ (1 \leq i \leq 4)$	$\mathcal{D}_i\ (1 \leq i \leq 4)$: (2a), $a_{\mathcal{D}_i} = 1$
G_2	$2\alpha'_i\ (1 \leq i \leq 2)$	$\mathcal{D}_i\ (1 \leq i \leq 2)$: (2a), $a_{\mathcal{D}_i} = 1$

Table 5.2: Spherical roots and type of colors of $O_{\mathfrak{g}}$.

5.4 Minimial Rational Curves on Hilbert Schemes

As a final corollary, we describe *minimal rational curves* on the smooth projective symmetric variety $\mathbf{H}_{\mathfrak{g}}^{nor}$ (Corollary 5.3.1), in terms of conics on $Z_{\mathfrak{g}}$. To do this, recall the following definition:

Definition 5.4.1. Let X be a smooth projective variety, and $\mathcal{K} \subset \mathrm{RatCurves}(X)$ (Subsection 2.2.2) an irreducible component.

1. $\mathcal{K}$ is called a *family of minimal rational curves* on X if for general $x \in X$, $\mathcal{K}_x$ is nonempty and projective.
2. Assume that $\mathcal{K}$ is a family of minimal rational curves. For general $x \in X$ and the rational map

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(T_x X), \quad [C] \mapsto [T_x C]$$

(defined over the locus of rational curves smooth at x), the closure of the image of τ_x is called the *variety of minimal rational tangents* (*VMRT* for short) of $\mathcal{K}$ at x.

A family of minimal rational curves exists on X if and only if X is uniruled ([20, Proposition II.2.14]). In particular, since spherical varieties are rational (this is because any B-orbit is rational; see [5, V.15.13.(a)]), a smooth projective spherical variety admits a family of minimal rational curves. Especially, minimal rational curves on symmetric varieties are studied by [8] and [9]. Let us recall a special case of their result:

Theorem 5.4.2 ([8], [9]; cf. Remark 5.4.3)**.** *Let G' be a connected simple Lie group acting on a smooth projective variety X. Suppose that X is G'-symmetric. Let X^0 be an open G'-orbit in X, $o' \in X^0$ a point, and $K' := \mathrm{Stab}_{G'}(o')$. Assume that K' is semi-simple. Then we have the following:*

1. *X admits a unique family $\mathcal{K}$ of minimal rational curves. Moreover, it has the following properties:*
   - *(a) $\mathcal{K}_{o'}$ consists of smooth rational curves.*
   - *(b) $\mathcal{K}_{o'}$ contains a unique closed orbit under the action of the identity component $(K')^0$, containing a rational curve $\overline{\exp(l) \cdot o'}$ for a highest weight line $l \subset T_{o'}X$.*
   - *(c) $\mathcal{K}_{o'}$ is smooth and connected.*
2. *If furthermore X has no color as a G'/K'-embedding, then the following hold:*
   - *(a) For the VMRT $\mathcal{C}_{o'}$ of $\mathcal{K}$ at o', the tangent map $\tau_{o'} : \mathcal{K}_{o'} \to \mathcal{C}_{o'}$ (Definition 5.4.1) is an isomorphism.*
   - *(b) If the restricted root system of G'/K' is not of type A, then $\mathcal{K}_{o'}$ is K'-homogeneous.*

Remark 5.4.3. The statements of Theorem 5.4.2 can be deduced as follows. Whenever K' is semi-simple, the isotropy representation of G'/K' is $(K')^0$-irreducible ([45, §8.12]). Thus the first statement follows from [8, Proposition 2.6], and the second statement follows from [9, Theorem 5.1–5.2].

Now we consider minimal rational curves on $\mathbf{H}_{\mathfrak{g}}^{nor}$, which is a smooth projective symmetric variety (Corollary 5.3.1). If $\mathfrak{g}$ is of type A or C, then we have a concrete description of $\mathbf{H}_{\mathfrak{g}}^{nor}$ (Subsections 3.1.1 and 5.1.1), hence its minimal rational curves can be easily described.

Therefore, from now on, we only consider the case where $\mathfrak{g}$ is not of type A or C. To apply Theorem 5.4.2, let $\mathbf{D}_{\mathfrak{g}}$ be a smooth projective $O_{\mathfrak{g}}$-embedding without color and equipped with a G-equivariant

$\mathfrak{g}$	$\mathcal{D}_{o'} \hookrightarrow \mathbb{P}(T_{o'}\mathbf{D}_{\mathfrak{g}})$
$B_r \ (r \geq 4)$	$\mathbb{Q}^2 \times \mathbb{Q}^{2r-5}$
$D_r \ (r \geq 5)$	$\mathbb{Q}^2 \times \mathbb{Q}^{2r-6}$
B_3	$\nu_2(\mathbb{P}^1) \times \mathbb{Q}^2$
D_4	$\mathbb{Q}^2 \times \mathbb{Q}^2$
E_6	$\mathrm{Gr}(3, 6) \times \mathbb{P}^1$
E_7	$\mathrm{OG}(6, 12) \times \mathbb{P}^1$
E_8	$E_7/P_1 \times \mathbb{P}^1$
F_4	$\mathrm{LG}(3, 6) \times \mathbb{P}^1$
G_2	$\mathbb{P}^1 \times \nu_3(\mathbb{P}^1)$

Table 5.3: VMRT $\mathcal{D}_{o'}$ of an $O_{\mathfrak{g}}$-embedding $\mathbf{D}_{\mathfrak{g}}$ without color.

birational morphism $\pi : \mathbf{D}_{\mathfrak{g}} \to \mathbf{H}_{\mathfrak{g}}^{nor}$. Such $\mathbf{D}_{\mathfrak{g}}$ exists, since, for example, one can take an equivariant resolution of singularities of the decoloration of $\mathbf{H}_{\mathfrak{g}}^{nor}$ and then apply the equivariant Chow lemma. Recall that $R'_{O_{\mathfrak{g}}}$ is not of type A unless $\mathfrak{g} = C_2$, and that G^{σ} is semi-simple unless $\mathfrak{g}$ is of type A (Theorem 3.2.1). Therefore $\mathbf{D}_{\mathfrak{g}}$ satisfies the assumptions in Theorem 5.4.2. Then the following corollary is a direct consequence of Theorem 5.4.2 and [9, Table 1].

Corollary 5.4.4. *Assume that $\mathfrak{g}$ is not of type A or C. Let $o' := [C_{\rho}] \in O_{\mathfrak{g}}$ (Lemma 3.3.3) be the base point. Suppose that $\mathbf{D}_{\mathfrak{g}}$ is a smooth projective $O_{\mathfrak{g}}$-embedding without color. Then $\mathbf{D}_{\mathfrak{g}}$ has a unique family $\mathcal{D}$ of minimal rational curves, and $\mathcal{D}_{o'}$ is G^{σ}-homogeneous and consisting of smooth rational curves. Its VMRT $\mathcal{D}_{o'} \hookrightarrow \mathbb{P}(T_{o'}\mathbf{D}_{\mathfrak{g}})$ at $o' \in O_{\mathfrak{g}}(\subset \mathbf{D}_{\mathfrak{g}})$ is described in Table 5.3.*

Now we state the main result of this section.

Theorem 5.4.5. *In the setting of Corollary 5.4.4, assume that there is a G-equivariant birational morphism $\pi : \mathbf{D}_{\mathfrak{g}} \to \mathbf{H}_{\mathfrak{g}}^{nor}$. Then $\mathbf{H}_{\mathfrak{g}}^{nor}$ admits a unique family $\mathcal{H}$ of minimal rational curves, and the morphism π induces an isomorphism $\pi_* : \mathcal{D}_{o'} \to \mathcal{H}_{o'}$, $[C] \mapsto [\pi(C)]$.*

To prove Theorem 5.4.5, we need a better understanding of minimal rational curves on $\mathbf{H}_{\mathfrak{g}}^{nor}$. From now on, assume that $\mathfrak{g}$ is not of type A or C, and let $\mathcal{H}$ be a (unique) family of minimal rational curves on $\mathbf{H}_{\mathfrak{g}}^{nor}$, which exists by Theorem 5.4.2. Then since

$$T_{o'}O_{\mathfrak{g}} \simeq \mathfrak{g}/\mathfrak{g}^{\sigma} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$$

as G^{σ}-representations, $\mathfrak{g}_{\rho-\alpha_{j_0}}$ is a highest weight line, where α_{j_0} is the unique neighbor of $-\rho$ in the extended Dynkin diagram of $\mathfrak{g}$ (Section 2.1). Thus the rational curve $\overline{\exp(\mathfrak{g}_{\rho-\alpha_{j_0}}) \cdot o'}$ is in the unique closed G^{σ}-orbit in $\mathcal{H}_{o'}$.

We describe the unique closed G^{σ}-orbit in $\mathcal{H}_{o'}$ in terms of conics on $Z_{\mathfrak{g}}$. Let x be a point in $C_{\rho}(\subset Z_{\mathfrak{g}})$. Recall that $\mathbb{P}(T_x Z_{\mathfrak{g}}) \setminus \mathbb{P}(D_x)$ parametrizes twistor conics passing through x on $Z_{\mathfrak{g}}$ (Theorem 3.3.4), and that the space $\mathcal{C}_x$ of lines through x is in the hyperplane $\mathbb{P}(D_x)$ (Subsection 2.2.2). For each $[l] \in \mathcal{C}_x$, consider the linear line joining $[T_x C_{\rho}] \in \mathbb{P}(T_x Z_{\mathfrak{g}}) \setminus \mathbb{P}(D_x)$ and $[l] \in \mathbb{P}(D_x)$, denoted by $\overline{o'l}$. Then every point in $\overline{o'l} \setminus \{[l]\}$ is lying in $\mathbb{P}(T_x Z_{\mathfrak{g}}) \setminus \mathbb{P}(D_x)$, hence corresponds to a twistor conic through x. Thus we may define

$$\mathcal{T}_{x,[l]} := \overline{\{[C] \in O_{\mathfrak{g}} : x \in C, [T_x C] \in \overline{o'l}\}}, \quad \forall x \in C_{\rho}, \quad [l] \in \mathcal{C}_x.$$

This is a rational curve on $\mathbf{H}^{nor}_{\mathfrak{g}}$, since its open part is isomorphic to $\overline{o'l} \setminus \{[l]\} \simeq \mathbb{C}^1$. In fact, we have

$$\mathcal{T}_{o,[\mathfrak{g}_{-\alpha_{j_0}}]} = \overline{\exp(\mathfrak{g}_{\rho-\alpha_{j_0}}) \cdot o'}, \tag{5.1}$$

since for $t \in \mathbb{C}$,

$$\exp(t \cdot E_{\rho-\alpha_{j_0}}) \cdot C_\rho = \exp(t \cdot E_{\rho-\alpha_{j_0}}) \cdot \overline{\exp(\mathfrak{g}_{-\rho}) \cdot o} = \overline{\exp(Ad_{t \cdot E_{\rho-\alpha_{j_0}}} \mathfrak{g}_{-\rho}) \cdot o}$$

represents a point $[E_{-\rho} + t \cdot N_{\rho-\alpha_{j_0}, -\rho} E_{-\alpha_{j_0}}]$ in $\mathbb{P}(T_o Z_{\mathfrak{g}})$, which is in the line joining $[T_o C_\rho]$ and $[\mathfrak{g}_{-\alpha_{j_0}}]$. Moreover, since the Lie algebra of the standard Levi subgroup L of P is $\mathfrak{g}_0$, L fixes both $o \in Z_{\mathfrak{g}}$ and $o' \in \mathbf{H}^{nor}_{\mathfrak{g}}$, and for any $g \in L$, we have

$$g \cdot \mathcal{T}_{o,[l]} = \mathcal{T}_{o, g\cdot[l]}, \quad \forall [l] \in \mathcal{C}_o.$$

Since L acts on $\mathcal{C}_o$ transitively (Subsection 2.2.2), we conclude that each $\mathcal{T}_{o,[l]}$ ($[l] \in \mathcal{C}_o$) represents a point in $\mathcal{H}_{o'}$, and that

$$\{[\mathcal{T}_{o,[l]}] \in \mathcal{H}_{o'} : [l] \in \mathcal{C}_o\}$$

is L-homogeneous. Since G^σ acts on C_ρ transitively and

$$k \cdot \mathcal{T}_{o,[l]} = \mathcal{T}_{k\cdot o, k\cdot[l]}, \quad \forall k \in G^\sigma,$$

we see that

$$\{[\mathcal{T}_{x,[l]}] \in \mathcal{H}_{o'} : x \in C_\rho, [l] \in \mathcal{C}_x\}$$

is G^σ-homogeneous. In fact, it is projective by the equation (5.1) and Theorem 5.4.2.

Lemma 5.4.6. *Keep the previous notation. For each $x \in C_\rho$ and $[l] \in \mathcal{C}_x$, $\mathcal{T}_{x,[l]} \setminus O_{\mathfrak{g}}$ consists of a single point t_∞ such that*

1. *t_∞ is represented by a non-planar reducible conic on $Z_{\mathfrak{g}}$, and*
2. *$G \cdot t_\infty$ is of codimension 1 in $\mathbf{H}^{nor}_{\mathfrak{g}}$.*

Remark 5.4.7. Lemma 5.4.6 means that t_∞ represents a general point of the prime divisor given in Proposition 5.2.2.

Proof of Lemma 5.4.6. Recall that $\mathcal{T}_{x,[l]} \setminus O_{\mathfrak{g}}$ is isomorphic to $\mathbb{C}^1$, hence its boundary is a single point t_∞. To show the statements on t_∞, by homogeneity, we may assume that $x = o$ and $l = \mathfrak{g}_{-\alpha_{j_0}} \mod \mathfrak{p}$. Then the members of $\mathcal{T}_{o,[l]} \setminus \{t_\infty\}$ can be written as for $t \in \mathbb{C}$,

$$\exp(t \cdot E_{\rho-\alpha_{j_0}}) \cdot C_\rho = Z_{\mathfrak{g}} \cap \mathbb{P}(Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_\rho, Ad_{t\cdot E_{\rho-\alpha_{j_0}}} H_\rho, Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_{-\rho}).$$

Since

$$\begin{aligned} Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_\rho &= E_\rho, \\ Ad_{t\cdot E_{\rho-\alpha_{j_0}}} H_\rho &= H_\rho - t \cdot \langle \rho, \rho - \alpha_{j_0} \rangle \cdot E_{\rho-\alpha_{j_0}}, \\ Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_{-\rho} &= E_{-\rho} + t \cdot N_{\rho-\alpha_{j_0}, -\rho} \cdot E_{-\alpha_{j_0}}, \end{aligned}$$

we have

$$\mathbb{P}(Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_\rho, Ad_{t\cdot E_{\rho-\alpha_{j_0}}} H_\rho, Ad_{t\cdot E_{\rho-\alpha_{j_0}}} E_{-\rho}) = \mathbb{P}(E_\rho, H_\rho - t\cdot\langle \rho, \rho-\alpha_{j_0}\rangle \cdot E_{\rho-\alpha_{j_0}}, E_{-\rho} + t\cdot N_{\rho-\alpha_{j_0}, -\rho} \cdot E_{-\alpha_{j_0}}).$$

By taking $t \to \infty$, we see that the limit curve is contained in

$$Z_{\mathfrak{g}} \cap \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}}).$$

As

$$\overline{\exp(\mathfrak{g}_{-\alpha_{j_0}}) \cdot [E_\rho]} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}})$$

(which is the line tangent to l at o) and

$$\overline{\exp(\mathfrak{g}_\rho) \cdot [E_{-\alpha_{j_0}}]} = \mathbb{P}(E_{-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}}),$$

the intersection contains a reducible conic singular at $[\mathfrak{g}_{\rho-\alpha_{j_0}}]$. In fact, since the line $\mathbb{P}(E_\rho, E_{-\alpha_{j_0}})$ is not contained in $Z_{\mathfrak{g}}$, we see that

$$t_\infty = [\mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}) \cup \mathbb{P}(E_{-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}})].$$

Now it is enough to show that the G-orbit of the reducible conic

$$\mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}) \cup \mathbb{P}(E_{-\alpha_{j_0}}, E_{\rho-\alpha_{j_0}})$$

is of dimension $(4n-1)$. In fact, since the reflection $s_{\alpha_{j_0}} \in W$ acts by

$$s_{\alpha_{j_0}}(\rho) = \rho - \langle \rho | \alpha_{j_0} \rangle \cdot \alpha_{j_0} = \rho - \alpha_{j_0}$$

and

$$s_{\alpha_{j_0}}(\alpha_{j_0}) = -\alpha_{j_0},$$

we may compute the isotropy group of the reducible conic

$$\mathbb{P}(E_{\rho-\alpha_{j_0}}, E_\rho) \cup \mathbb{P}(E_{\alpha_{j_0}}, E_\rho).$$

Since its singular point is $o = [E_\rho]$, the identity component of the isotropy group is equal to

$$\{g \in P : Ad_g \mathfrak{g}_{-\alpha_{j_0}} \equiv \mathfrak{g}_{-\alpha_j} \mod \mathfrak{p}, \quad Ad_g \mathfrak{g}_{\alpha_{j_0}-\rho} \equiv \mathfrak{g}_{\alpha_j-\rho} \mod \mathfrak{p}\}.$$

Hence its Lie algebra is

$$\left\{X \in \mathfrak{p} : [X, \mathfrak{g}_{-\alpha_{j_0}}] \leq \mathfrak{g}_{-\alpha_{j_0}} \mod \mathfrak{p}, \quad [X, \mathfrak{g}_{\alpha_{j_0}-\rho}] \leq \mathfrak{g}_{\alpha_{j_0}-\rho} \mod \mathfrak{p}\right\}.$$

Observe that

$$\begin{aligned}
\left\{X \in \mathfrak{p} : [X, \mathfrak{g}_{-\alpha_{j_0}}] \leq \mathfrak{g}_{-\alpha_{j_0}} \mod \mathfrak{p}\right\} &= \mathfrak{h} \oplus \bigoplus_{\alpha<0,\, m_{j_0}(\alpha)=0,\, \alpha-\alpha_{j_0} \notin R} \mathfrak{g}_\alpha \\
&= \mathfrak{h} \oplus \bigoplus_{\alpha<0,\, m_{j_0}(\alpha)=0,\, \langle \alpha, \alpha_{j_0} \rangle=0} \mathfrak{g}_\alpha, \\
\left\{X \in \mathfrak{p} : [X, \mathfrak{g}_{\alpha_{j_0}-\rho}] \leq \mathfrak{g}_{\alpha_{j_0}-\rho} \mod \mathfrak{p}\right\} &= \mathfrak{t} \oplus \bigoplus_{\alpha<0,\, m_{j_0}(\alpha)=0} \mathfrak{g}_\alpha \oplus \mathfrak{g}^1 \oplus \bigoplus_{\alpha>0,\, m_{j_0}(\alpha)=0,\, \alpha+\alpha_{j_0}-\rho \notin R} \mathfrak{g}_\alpha \\
&= \mathfrak{t} \oplus \bigoplus_{\alpha<0,\, m_{j_0}(\alpha)=0} \mathfrak{g}_\alpha \oplus \mathfrak{g}^1 \oplus \bigoplus_{\alpha>0,\, m_{j_0}(\alpha)=0,\, \langle \alpha, \alpha_{j_0}-\rho \rangle=0} \mathfrak{g}_\alpha \\
&= \mathfrak{t} \oplus \bigoplus_{\alpha<0,\, m_{j_0}(\alpha)=0} \mathfrak{g}_\alpha \oplus \mathfrak{g}^1 \oplus \bigoplus_{\alpha>0,\, m_{j_0}(\alpha)=0,\, \langle \alpha, \alpha_{j_0} \rangle=0} \mathfrak{g}_\alpha.
\end{aligned}$$

(For the second part, note that $\alpha_{j_0} - \rho$ is the minimum in $R \setminus \{-\rho\}$). Therefore the Lie algebra of the isotropy group is

$$\mathfrak{t} \oplus \bigoplus_{m_{j_0}(\alpha)=0,\, \langle \alpha, \alpha_{j_0} \rangle=0} \mathfrak{g}_\alpha \oplus \mathfrak{g}^1.$$

In other words, the tangent space of the G-orbit of the reducible conic can be identified with

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \bigoplus_{m_{j_0}(\alpha)=0, \langle \alpha, \alpha_{j_0} \rangle \neq 0} \mathfrak{g}_\alpha.$$

Its dimension is

$$\dim Z_{\mathfrak{g}} + \#(\text{roots of } P_{\alpha_{j_0}}) - \#(\text{root of } P_{\alpha_{j_0}, N(\alpha_{j_0})}).$$

Since

$$\#(\text{roots of } P_{\alpha_{j_0}}) = \begin{cases} |R_{A_1 \times B_{r-2}}| = 2 + 2(r-2)^2 & (\mathfrak{g} = B_r,\, r \geq 4) \\ |R_{A_1 \times A_1}| = 2 + 2 & (\mathfrak{g} = B_3) \\ |R_{A_1 \times D_{r-2}}| = 2 + 2(r-2)(r-3) & (\mathfrak{g} = D_r,\, r \geq 5) \\ |R_{A_1 \times A_1 \times A_1}| = 2 + 2 + 2 & (\mathfrak{g} = D_4) \\ |R_{A_5}| = 30 & (\mathfrak{g} = E_6) \\ |R_{D_6}| = 60 & (\mathfrak{g} = E_7) \\ |R_{E_7}| = 126 & (\mathfrak{g} = E_8) \\ |R_{C_3}| = 18 & (\mathfrak{g} = F_4) \\ |R_{A_1}| = 2 & (\mathfrak{g} = G_2) \end{cases}$$

and

$$\#(\text{roots of } P_{\alpha_{j_0}, N(\alpha_{j_0})}) = \begin{cases} |R_{B_{r-3}}| = 2(r-3)^2 & (\mathfrak{g} = B_r,\, r \geq 5) \\ |R_{A_1}| = 2 & (\mathfrak{g} = B_4) \\ 0 & (\mathfrak{g} = B_3) \\ |R_{D_{r-3}}| = 2(r-3)(r-4) & (\mathfrak{g} = D_r,\, r \geq 6) \\ |R_{A_1 \times A_1}| = 2 + 2 & (\mathfrak{g} = D_5) \\ 0 & (\mathfrak{g} = D_4) \\ |R_{A_2 \times A_2}| = 6 + 6 & (\mathfrak{g} = E_6) \\ |R_{A_5}| = 30 & (\mathfrak{g} = E_7) \\ |R_{E_6}| = 72 & (\mathfrak{g} = E_8) \\ |R_{A_2}| = 6 & (\mathfrak{g} = F_4) \\ 0 & (\mathfrak{g} = G_2) \end{cases}$$

we see that

$$\#(\text{roots of } P_{\alpha_{j_0}}) - \#(\text{root of } P_{\alpha_{j_0}, N(\alpha_{j_0})}) = \begin{cases} 4r - 8 & (\mathfrak{g} = B_r,\, r \geq 3) \\ 4r - 10 & (\mathfrak{g} = D_r,\, r \geq 4) \\ 18 & (\mathfrak{g} = E_6) \\ 30 & (\mathfrak{g} = E_7) \\ 54 & (\mathfrak{g} = E_8) \\ 12 & (\mathfrak{g} = F_4) \\ 2 & (\mathfrak{g} = G_2) \end{cases} = 2n - 2.$$

Therefore

$$\dim Z_{\mathfrak{g}} + \#(\text{roots of } P_{\alpha_{j_0}}) - \#(\text{root of } P_{\alpha_{j_0}, N(\alpha_{j_0})}) = 4n - 1.$$

□

Proof of Theorem 5.4.5. $\mathbf{H}_{\mathfrak{g}}^{nor}$ admits exactly one family $\mathcal{H}$ of minimal rational curves and $\mathcal{H}_o$ is irreducible by Theorem 3.2.1 and Theorem 5.4.2. Since π is a birational morphism, π induces a G^σ-equivariant embedding $\pi_* : \mathcal{D}_o \hookrightarrow \mathcal{H}_o$ by [9, Lemma 2.4 and Remark 2.5]. Since $\mathcal{D}_o$ is G^σ-homogeneous and $\mathcal{H}_o$ is irreducible, to show that π_* is surjective, it is enough to show that $\dim \mathcal{D}_o = \dim \mathcal{H}_o$. To do

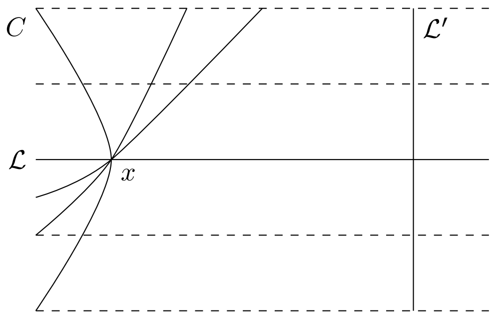


Figure 5.9: Locus in $Z_{\mathfrak{g}}$ swept by conics parametrized by a minimal rational curve on $\mathbf{H}_{\mathfrak{g}}^{nor}$.

this, recall that for a rational curve $[C] \in \mathcal{D}_o$, and its image $[\pi(C)] \in \mathcal{H}_o$, both C and $\pi(C)$ are free, hence we have

$$\dim \mathcal{D}_o = \deg_C K_{\mathbf{D}_{\mathfrak{g}}}^{-1} - 2, \quad \text{and} \quad \dim \mathcal{H}_o = \deg_{\pi(C)} K_{\mathbf{H}_{\mathfrak{g}}^{nor}}^{-1} - 2.$$

(See [20, Theorems II.1.7 and II.2.16].) Thus we need to show that C and $\pi(C)$ have the same anti-canonical degree. In fact, by Lemma 5.4.6, if $t_\infty \in \pi(C) \setminus O_{\mathfrak{g}}$, then $G \cdot t_\infty$ is of codimension 1, hence the birational morphism $\pi : \mathbf{D}_{\mathfrak{g}} \to \mathbf{H}_{\mathfrak{g}}^{nor}$ is an isomorphism over $U := O_{\mathfrak{g}} \cup (G \cdot t_\infty)$ (which is an open subset of $\mathbf{H}_{\mathfrak{g}}^{nor}$ since there are only finitely many G-orbits). Since $\pi(C) \subset U$, C and $\pi(C)$ have the same anti-canonical degree. Hence the statement follows. □

Remark 5.4.8. Theorem 5.4.5, together with Table 5.3 and Table 2.2, shows that the VMRT of $\mathbf{H}_{\mathfrak{g}}^{nor}$ is isomorphic to $\mathbb{P}^1 \times \mathcal{C}_o$. This reflects the fact that the VMRT of $\mathbf{H}_{\mathfrak{g}}^{nor}$ at $[C_\rho]$ consists of $[T_{o'}(\mathcal{T}_{x,[l]})]$ for $x \in C_\rho (\simeq \mathbb{P}^1)$ and $[l] \in \mathcal{C}_o$.

Finally, we describe the locus swept by conics parametrized by $\mathcal{T}_{x,[l]}$ in $Z_{\mathfrak{g}}$.

Proposition 5.4.9. *Assume that $\mathfrak{g}$ is not of type A or C. Let x be a point in $Z_{\mathfrak{g}}$, C a twistor conic, and $\mathcal{L}$ a line on $Z_{\mathfrak{g}}$ such that $x \in C \cap \mathcal{L}$. Put $l := T_x\mathcal{L}$, and let $\mathfrak{a}$ be the smallest Lie subalgebra of $\mathfrak{g}$ such that $C \cup \mathcal{L} \subset \mathbb{P}(\mathfrak{a})$.*

1. $\dim \mathfrak{a} = 5$*, and the $\mathfrak{sl}_2$-algebra generated by C is a maximal reductive subalgebra of $\mathfrak{a}$. In particular, the unipotent radical $\mathfrak{u}$ of $\mathfrak{a}$ is of dimension 2.*

2. *The line $\mathcal{L}' := \mathbb{P}(\mathfrak{u})$ does not intersect with the plane spanned by C, and $[\mathcal{L} \cup \mathcal{L}']$ is the unique boundary point $\mathcal{T}_{x,[l]} \setminus O_{\mathfrak{g}}$.*

3. *The intersection $\mathbb{P}(\mathfrak{a}) \cap Z_{\mathfrak{g}}$ is the union of conics parametrized by $\mathcal{T}_{x,[l]}$.*

4. $\mathbb{P}(\mathfrak{a}) \cap Z_{\mathfrak{g}}$ *is a cubic scroll in $\mathbb{P}(\mathfrak{a})$ with its directrix $\mathcal{L}'$, and $\mathcal{L}$ is a line of the ruling. More precisely,*

$$\mathbb{P}(\mathfrak{a}) \cap Z_{\mathfrak{g}} = \bigcup_{s \in \mathcal{L}'} \mathbb{P}(s, f(s))$$

for an isomorphism $f : \mathcal{L}' \to C$ such that $\mathcal{L} = \mathbb{P}(s, f(s))$ for some $s \in \mathcal{L}'$.

Proof. As before, we may assume that

$$x = o(= [E_\rho]), \quad C = C_\rho(= \mathbb{P}(E_\rho, H_\rho, E_{-\rho}) \cap Z_{\mathfrak{g}}), \quad \mathcal{L} = \mathbb{P}(E_\rho, E_{\rho-\alpha_{j_0}}) \quad (l = \mathfrak{g}_{-\alpha_{j_0}} \mod \mathfrak{p}).$$

Then

$$\mathfrak{a} = \mathbb{C}\langle E_\rho, H_\rho, E_{-\rho}, E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}} \rangle$$

and its unipotent radical $\mathfrak{u}$ is $\mathbb{C}\langle E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}}\rangle$. Now the first two statements follow by the proof of Lemma 5.4.6, since $\mathcal{L}' = \mathbb{P}(E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}})$.

To show the third statement, consider the homogeneous coordinate

$$[x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}] \in \mathbb{P}(\mathfrak{a}).$$

If a point in $\mathbb{P}(\mathfrak{a})$ is contained in $Z_{\mathfrak{g}}$, it satisfies the relations

$$\langle x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}, x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}\rangle = 0 \quad (5.2)$$

and

$$(ad_{x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}})^3(E_{-\rho}) = 0. \quad (5.3)$$

To simplify the notation, we define

$$c := \frac{\langle\rho,\rho\rangle}{2} = \langle\rho, \alpha_{j_0}\rangle = \langle\rho, \rho-\alpha_{j_0}\rangle \quad \text{and} \quad N := N_{\rho-\alpha_{j_0}, -\rho} = N_{-\rho, \alpha_{j_0}} = N_{\alpha_{j_0}, \rho-\alpha_{j_0}}$$

(see [14, Lemma 5.1]). Then by [14, Theorem 5.5], we can choose the root vectors so that

$$N^2 = c \neq 0.$$

Now the equation (5.2) reads

$$y^2c + xz = 0.$$

On the other hand, for the equation (5.3), we have

$$[x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}, E_{-\rho}] = x\cdot H_\rho + y(-2c)\cdot E_{-\rho} + uN\cdot E_{-\alpha_{j_0}},$$

$$\begin{aligned}
&(ad_{x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}})^2(E_{-\rho})\\
=&[x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}, x\cdot H_\rho + y(-2c)\cdot E_{-\rho} + uN\cdot E_{-\alpha_{j_0}}]\\
=&x^2(-2c)\cdot E_\rho + xy(-2c)\cdot H_\rho + xuN(-N)\cdot E_{\rho-\alpha_{j_0}}\\
&+ y^2(-2c)^2\cdot E_{-\rho} + yuN(-c)\cdot E_{-\alpha_{j_0}}\\
&+ zx(2c)\cdot E_{-\rho}\\
&+ ux(-c)\cdot E_{\rho-\alpha_{j_0}} + uy(-2c)N\cdot E_{-\alpha_{j_0}}\\
&+ wxc\cdot E_{-\alpha_{j_0}}\\
=&x^2(-2c)\cdot E_\rho + xy(-2c)\cdot H_\rho + (2c)(y^2 2c + zx)\cdot E_{-\rho} + xu(-2c)\cdot E_{\rho-\alpha_{j_0}} + c(-3yuN + xw)\cdot E_{-\alpha_{j_0}}
\end{aligned}$$

and so

$$\begin{aligned}
&(ad_{x\cdot E_\rho + y\cdot H_\rho + z\cdot E_{-\rho} + u\cdot E_{\rho-\alpha_{j_0}} + w\cdot E_{-\alpha_{j_0}}})^3(E_{-\rho})\\
=&x^2y(2c)^2\cdot E_\rho + x(2c)(y^2 2c + zx)\cdot H_\rho + xc(-3yuN + xw)(-N)\cdot E_{\rho-\alpha_{j_0}}\\
&+ yx^2(-2c)(2c)\cdot E_\rho + (2c)(-2c)y(y^2 2c + zx)\cdot E_{-\rho} + yxu(-2c^2)\cdot E_{\rho-\alpha_{j_0}} + (-c^2)y(-3yuN + xw)\cdot E_{-\alpha_{j_0}}\\
&+ zx^2(2c)\cdot H_\rho + zxy(-2c)(2c)\cdot E_{-\rho} + zxu(-2c)(-N)\cdot E_{-\alpha_{j_0}}\\
&+ uxy(2c^2)\cdot E_{\rho-\alpha_{j_0}} + (2cN)u(y^2 2c + zx)\cdot E_{-\alpha_{j_0}}\\
&+ wx^2(-2c)N\cdot E_{\rho-\alpha_{j_0}} + wxy(-2c^2)\cdot E_{-\alpha_{j_0}}\\
=&3xcN(yuN - xw)\cdot E_{\rho-\alpha_{j_0}} + cN(7y^2uc - 3xywN + 4xzu)\cdot E_{-\alpha_{j_0}} \quad (\because y^2c + xz = 0).
\end{aligned}$$

Therefore $Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a})$ is contained in the locus of

$$y^2c + zx = 0, \quad x(yuN - xw) = 0, \quad 7y^2uc - 3xywN + 4xzu = 0.$$

If $x = 0$, then the equations imply $y^2 = 0$, i.e.

$$(x = 0) \cap Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) \subset \mathbb{P}(E_{-\rho}, E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}}).$$

Since

$$\mathbb{P}(E_{-\rho}, E_{-\alpha_{j_0}}) \cup \mathbb{P}(E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}}) \subset Z_{\mathfrak{g}}$$

but $\mathbb{P}(E_{-\rho}, E_{\rho-\alpha_{j_0}}) = s_\rho(\mathbb{P}(E_\rho, E_{-\alpha_{j_0}})) \notin Z_{\mathfrak{g}}$, we conclude that

$$(x = 0) \cap Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) = \mathbb{P}(E_{-\rho}, E_{-\alpha_{j_0}}) \cup \mathbb{P}(E_{\rho-\alpha_{j_0}}, E_{-\alpha_{j_0}}) = \mathcal{L}'' \cup \mathcal{L}'$$

where $\mathcal{L}'' := \mathbb{P}(E_{-\rho}, E_{-\alpha_{j_0}})$. If $x \neq 0$, then on the affine open subset $(x = 1)$, we have

$$z = -y^2c, \quad w = yuN \quad (7y^2uc - 3ywN + 4zu = 0).$$

Thus

$$(x \neq 0) \cap Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) \subset \left\{[E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho} + u \cdot E_{\rho-\alpha_{j_0}} + (yuN) \cdot E_{-\alpha_{j_0}}] : y, u \in \mathbb{C}\right\}.$$

For $y, u \in \mathbb{C}$, if $y \neq 0$, then

$$\exp(\frac{-u}{yc} \cdot E_{\rho-\alpha_{j_0}}) \cdot [E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho}] = [E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho} + u \cdot E_{\rho-\alpha_{j_0}} + (yuN) \cdot E_{-\alpha_{j_0}}],$$

which implies that

$$\left\{[E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho} + u \cdot E_{\rho-\alpha_{j_0}} + (yuN) \cdot E_{-\alpha_{j_0}}] : y \in \mathbb{C}^\times, u \in \mathbb{C}\right\} \subset \bigcup_{[C'] \in \mathcal{T}_{o,[l]} \cap O_{\mathfrak{g}}} C'.$$

In fact, for $[C'] \in \mathcal{T}_{o,[l]} \cap O_{\mathfrak{g}}$, every element in $C' \smallsetminus \mathcal{L}''$ can be written in form

$$\exp(\frac{-u}{yc} \cdot E_{\rho-\alpha_{j_0}}) \cdot [E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho}]$$

for some $0 \neq y \in \mathbb{C}$ and $u \in \mathbb{C}$, hence we have

$$\left\{[E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho} + u \cdot E_{\rho-\alpha_{j_0}} + (yuN) \cdot E_{-\alpha_{j_0}}] : y \in \mathbb{C}^\times, u \in \mathbb{C}\right\} = \bigcup_{[C'] \in \mathcal{T}_{o,[l]} \cap O_{\mathfrak{g}}} C' \smallsetminus \mathcal{L}'',$$

and it is contained in $Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a})$. If $y = 0$, then

$$[E_\rho + y \cdot H_\rho + (-y^2c) \cdot E_{-\rho} + u \cdot E_{\rho-\alpha_{j_0}} + (yuN) \cdot E_{-\alpha_{j_0}}] = [E_\rho + u \cdot E_{\rho-\alpha_{j_0}}] \in \mathcal{L},$$

hence

$$(x \neq 0) \cap Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) = \bigcup_{[C'] \in \mathcal{T}_{o,[l]} \cap O_{\mathfrak{g}}} (C' \smallsetminus \mathcal{L}'') \cup \mathcal{L}.$$

Therefore

$$Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) = \bigcup_{[C'] \in \mathcal{T}_{o,[l]} \cap O_{\mathfrak{g}}} C' \cup \mathcal{L} \cup \mathcal{L}' = \bigcup_{[C'] \in \mathcal{T}_{o,[l]}} C'.$$

It is remained to describe $\mathbb{P}(\mathfrak{a}) \cap Z_{\mathfrak{g}}$ as a cubic scroll. Consider an isomorphism

$$f : \mathcal{L}' \to C_\rho, \quad [s_1 \cdot E_{\rho-\alpha_{j_0}} + s_2 N \cdot E_{-\alpha_{j_0}}] \mapsto [s_1^2 \cdot E_\rho + s_1 s_2 \cdot H_\rho + (-s_2^2 c) \cdot E_{-\rho}].$$

Then for $s = [s_1 \cdot E_{\rho-\alpha_{j_0}} + s_2 N \cdot E_{-\alpha_{j_0}}]$, we have

$$\mathcal{L}_s := \mathbb{P}(s,\, f(s)) = \begin{cases} \mathbb{P}(E_{\rho-\alpha_{j_0}} + yN \cdot E_{-\alpha_{j_0}},\, E_\rho + y \cdot H_\rho + (-y^2 c) \cdot E_{-\rho}) & (s_1 \neq 0,\, y := s_2/s_1) \\ \mathbb{P}(E_{-\alpha_{j_0}},\, E_{-\rho}) (= \mathcal{L}'') & (s_1 = 0). \end{cases}$$

In particular, $\mathcal{L}_{[E_{\rho-\alpha_{j_0}}]} = \mathcal{L}$ and $\mathcal{L}_{[E_{-\alpha_{j_0}}]} = \mathcal{L}''$. Thus

$$\bigcup_{s \in \mathcal{L}'} \mathcal{L}_s \cap (x = 0) = \mathcal{L}' \cup \mathcal{L}'' = Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) \cap (x = 0)$$

and

$$\bigcup_{s \in \mathcal{L}'} \mathcal{L}_s \cap (x \neq 0) = \bigcup_{y \in \mathbb{C}} \mathcal{L}_{[E_{\rho-\alpha_{j_0}} + y \cdot E_{-\alpha_{j_0}}]} \cap (x \neq 0) = Z_{\mathfrak{g}} \cap \mathbb{P}(\mathfrak{a}) \cap (x \neq 0).$$

□

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